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# FINITE-TIME STABILIZATION OF SYSTEMS OF CONSERVATION LAWS ON NETWORKS

VINCENT PERROLLAZ AND LIONEL ROSIER

**ABSTRACT.** We investigate the finite-time boundary stabilization of a 1-D first order quasilinear hyperbolic system of diagonal form on  $[0,1]$ . The dynamics of both boundary controls are governed by a finite-time stable ODE. The solutions of the closed-loop system issuing from small initial data in  $\text{Lip}([0,1])$  are shown to exist for all times and to reach the null equilibrium state in finite time. When only one boundary feedback law is available, a finite-time stabilization is shown to occur roughly in a twice longer time. The above feedback strategy is then applied to the Saint-Venant system for the regulation of water flows in a network of canals.

## 1. INTRODUCTION

Solutions of certain asymptotically stable ODE may reach the equilibrium state in finite time. This phenomenon, which is common when using feedback laws that are not Lipschitz continuous, was termed *finite-time stability* in [5] and investigated in that paper.

A finite-time stabilizer is a feedback control for which the closed-loop system is finite-time stable around some equilibrium. In some sense, it satisfies a controllability objective with a control in feedback form. On the other hand, a finite-time stabilizer may be seen as an exponential stabilizer yielding an arbitrarily large decay rate for the solutions to the closed-loop system. This explains why some efforts were made in the last decade to construct finite-time stabilizers for controllable systems, including the linear ones. See [29, 30] for some recent developments and up-to-date references, and [2] for some connections with Lyapunov theory.

For PDEs, the relationship between exact controllability and rapid stabilization was investigated in [34, 22, 23]. (See also [24] for the rapid semiglobal stabilization of the Korteweg-de Vries equation using a time-varying feedback law.)

To the best knowledge of the authors, the analysis of the finite-time stabilization of PDE is not developed yet. However, the phenomenon of finite-time extinction exists naturally for certain nonlinear evolution equations (see [36, 11, 6]). On the other hand, it is well-known since [28] that solutions of the wave equation on a bounded domain may disappear when using

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“transparent” boundary conditions. For instance, the solution of the 1-D wave equation

$$\partial_t^2 y - \partial_x^2 y = 0, \quad \text{in } (0, T) \times (0, 1), \quad (1.1)$$

$$\partial_x y(t, 1) = -\partial_t y(t, 1), \quad \text{in } (0, T), \quad (1.2)$$

$$\partial_x y(t, 0) = \partial_t y(t, 1), \quad \text{in } (0, T), \quad (1.3)$$

$$(y(0, \cdot), \partial_t y(0, \cdot)) = (y_0, z_0), \quad \text{in } (0, 1), \quad (1.4)$$

is finite-time stable in  $\{(y, z) \in H^1(0, 1) \times L^2(0, 1); y(0) + y(1) + \int_0^1 z(x) dx = 0\}$ , with  $T = 1$  as extinction time (see e.g. [23, Theorem 0.5] for the details.) The condition (1.2) is transparent in the sense that a wave  $y(t, x) = f(x - t)$  traveling to the right satisfies (1.2) and leaves the domain at  $x = 1$  without generating any reflected wave. Note that we can replace (1.3) by the boundary condition  $y(t, 0) = 0$  (or  $\partial_x y(t, 0) = 0$ ). Then a finite-time extinction still occurs (despite the fact that waves bounce at  $x = 0$ ) with an extinction time  $T = 2$ . We refer to [8] for the analysis of the finite-time extinction property for a nonhomogeneous string with a viscous damping at one extremity, and to [1] for the investigation of the finite-time stabilization of a network of strings.

The finite-time stability of (1.1)-(1.4) is easily established when writing (1.1) as a first order hyperbolic system

$$\partial_t \begin{pmatrix} r \\ s \end{pmatrix} - \partial_x \begin{pmatrix} s \\ r \end{pmatrix} = 0$$

with  $(r, s) = (\partial_x y, \partial_t y)$ , and next introducing the Riemann invariants  $u = r - s$ ,  $v = r + s$  that solve the system of two transport equations

$$\begin{aligned} \partial_t u + \partial_x u &= 0, \\ \partial_t v - \partial_x v &= 0. \end{aligned}$$

The boundary conditions (1.2) and (1.3) yield  $u(t, 0) = v(t, 1) = 0$  (and hence  $u(t, \cdot) = v(t, \cdot) = 0$  for  $t \geq 1$ ), while the boundary conditions (1.2) and  $y(t, 0) = 0$  yield  $v(t, 1) = 0$  and  $u(t, 0) = v(t, 0)$  (and hence  $v(t, \cdot) = 0$  for  $t \geq 1$  and  $u(t, \cdot) = 0$  for  $t \geq 2$ ).

The goal of this paper is to show that the finite-time extinction property can be realized for 1-D first order quasilinear hyperbolic systems

$$\partial_t Y + \partial_x F(Y) = 0, \quad (1.5)$$

that can be put in diagonal form, i.e. for which there is a smooth change of (dependent) variables that transforms (1.5) into a system of two nonlinear transport equations of the form

$$\partial_t u + \lambda(u, v) \partial_x u = 0, \quad (1.6)$$

$$\partial_t v + \mu(u, v) \partial_x v = 0, \quad (1.7)$$

where  $\mu(u, v) \leq -c < c \leq \lambda(u, v)$  are smooth functions and  $c > 0$  is some constant. In practice, the functions  $u$  and  $v$  are Riemann invariants of (1.5) (see e.g. [13]).

The generalization of the finite-time extinction property of the wave equation to systems of the form (1.6)-(1.7) is the main aim of this paper. Of course, one could just consider homogeneous Dirichlet conditions

$$u(t, 0) = v(t, 1) = 0,$$

but this would impose to restrict ourselves to initial data  $(u_0, v_0)$  fulfilling the compatibility conditions

$$u_0(0) = v_0(1) = 0.$$

Rather, we shall consider boundary conditions whose dynamics obey a finite-time stable ODE, namely

$$\frac{d}{dt}u(t, 0) = -K \operatorname{sgn}(u(t, 0))|u(t, 0)|^\gamma, \quad (1.8)$$

$$\frac{d}{dt}v(t, 1) = -K \operatorname{sgn}(v(t, 1))|v(t, 1)|^\gamma, \quad (1.9)$$

$((K, \gamma) \in (0, +\infty) \times (0, 1)$  being some constants) and supplement the system (1.6)-(1.7), (1.8)-(1.9) with the initial condition

$$u(0, x) = u_0(x), \quad v(0, x) = v_0(x). \quad (1.10)$$

The first main result in this paper (Theorem 1) asserts that for any pair  $(u_0, v_0)$  of (small enough) Lipschitz continuous initial data, system (1.6)-(1.7) and (1.8)-(1.10) admits a unique solution in some class of Lipschitz continuous functions, and that this solution is defined for all times  $t \geq 0$  and vanishes for roughly  $t \geq 1/c$ . Theorem 1 is proved by using a fixed-point argument (Schauder Theorem) and energy estimates.

Sometimes, the boundary condition at one extremity of the domain (say 0) is imposed by the context, so that we cannot chose the condition  $u(t, 0) = 0$  (or its generalization (1.8)) for the Riemann invariant  $u$ . Then, we have to replace (1.8) by a boundary condition of the form

$$u(t, 0) = h(v(t, 0), t), \quad (1.11)$$

for some (smooth) function  $h = h(v, t)$ . The second main result in this paper (Theorem 2) asserts that the system (1.6)-(1.7) and (1.9)-(1.11) is still locally well-posed with roughly an extinction time  $T = 2/c$ . The result is obtained for small initial data and for  $\|\partial_t h\|_\infty$  small enough.

The results obtained in this paper can be applied to:

- (1) the  $p$ -system

$$\partial_t r - \partial_x s = 0, \quad (1.12)$$

$$\partial_t s - \partial_x [p(r)] = 0, \quad (1.13)$$

where  $p \in C^1(\mathbb{R})$  is any given function;

- (2) the shallow water equations (also called Saint-Venant equations [33])

$$\partial_t H + \partial_x (HV) = 0, \quad (1.14)$$

$$\partial_t V + \partial_x \left( \frac{V^2}{2} + gH \right) = 0, \quad (1.15)$$

where  $H$  is the water depth and  $V(t, x)$  the averaged horizontal velocity of water in a canal, and  $g$  the gravitation constant;

- (3) Euler's equations for barotropic compressible gas

$$\partial_t \rho + \partial_x (\rho V) = 0, \quad (1.16)$$

$$\partial_t (\rho V) + \partial_x (\rho V^2 + p) = 0, \quad (1.17)$$

where  $\rho$  is the mass density,  $V$  the velocity, and  $p = p(\rho)$  the pressure of the gas.

- (4) The same strategy could in theory be applied to any system possessing Riemann invariants. Riemann invariants exist for most  $2 \times 2$  systems, and also for some larger systems (e.g. the  $3 \times 3$  system of Euler's equations for compressible gas, see [35, chapters 18,20]).

For the sake of shortness, we will limit ourselves to the stabilization of Saint-Venant equations, and will give an extension of the above finite-time stabilization results to a tree-shaped network of canals. The obtained extinction time will be roughly  $d/c$ , where  $d$  denotes the depth of the tree (Theorem 5).

There is a huge literature about the controllability and stabilization of first order hyperbolic equations (see e.g. [15, 14, 26, 31, 27, 16, 19]). In particular, the control of Saint-Venant equations has attracted the attention of the control community because of its relevance to the regulation of water flows in networks of canals or rivers. We refer the reader to e.g. [7, 37, 25, 20, 17, 4, 12, 18, 3], where Riemann invariants played often a great role in the design of the controls. Our main contribution here is to notice that a finite-time stabilization can be achieved as well, i.e. that bounces of waves at the two ends of the domain can be avoided.

A numerical scheme and some numerical experiments for the finite-time stabilization of water flows in a canal may be found in [32], in which certain results of this paper were announced.

The paper is outlined as follows. Classical but important properties of linear transport equations are recalled in Section 2. In Section 3, we introduce two boundary controls whose dynamics are governed by a finite-time stable ODE, and prove the existence and uniqueness of a solution to the closed-loop system, and the fact that this solution reaches the null state in finite time. In Section 4, we investigate the same problem with only one boundary control, the other boundary condition being imposed by the physical context. In the last section, we apply the results in Sections 3 and 4 to the regulation of water flows in a canal with one or two boundary controls, and extend the finite-time stabilization results to any tree-shaped network of canals.

## 2. SOME BACKGROUND ABOUT LINEAR TRANSPORT EQUATIONS

**2.1. Notations.**  $\mathcal{C}^0([0, T] \times [0, 1])$  denotes the space of continuous functions  $u : [0, T] \times [0, 1] \rightarrow \mathbb{R}$ . It is endowed with the norm

$$\|u\|_{\mathcal{C}^0([0, T] \times [0, 1])} = \sup_{(t, x) \in [0, T] \times [0, 1]} |u(t, x)|.$$

The norm of the space  $L^p(0, 1)$  is denoted  $\|\cdot\|_p$  for  $1 \leq p \leq \infty$ .  $\text{Lip}([0, 1])$  denotes the space of Lipschitz continuous functions  $u : [0, 1] \rightarrow \mathbb{R}$ . It may be identified with the Sobolev space  $W^{1, \infty}(0, 1)$ .  $\text{Lip}([0, 1])$  is endowed with the  $W^{1, \infty}(0, 1)$ -norm; that is

$$\|u\|_{\text{Lip}([0, 1])} = \|u\|_{W^{1, \infty}(0, 1)} = \|u\|_{\infty} + \|u'\|_{\infty}.$$

We use similar norms for  $\text{Lip}(\mathbb{R})$ ,  $\text{Lip}([0, T] \times [0, 1])$ , etc.

**2.2. Linear transport equation.** In this section we consider the initial boundary-value problem for the following linear transport equation

$$\partial_t y + a(t, x) \partial_x y = 0. \tag{2.1}$$

We assume thereafter that

$$a \in \mathcal{C}^0([0, T] \times [0, 1]) \cap L^\infty(0, T; \text{Lip}([0, 1])), \quad (2.2)$$

$$a(t, x) \geq c > 0, \quad \forall (t, x) \in [0, T] \times [0, 1], \quad (2.3)$$

where  $c$  denotes some constant. Note that the case when  $a(t, x) \leq -c < 0$  can be reduced to (2.3) by the transformation  $x \rightarrow 1 - x$ .

**2.3. Properties of the flow.** By (2.2),  $a$  is uniformly Lipschitz continuous in the variable  $x$ , with say a Lipschitz constant  $L = \|a\|_{L^\infty(0, T; \text{Lip}([0, 1]))}$ . Since we intend to use the method of characteristics to solve (2.1), we need to study the flow associated with  $a$ .

**Definition 1.** For  $(t, x) \in [0, T] \times [0, 1]$ , let  $\phi(\cdot, t, x)$  denote the  $\mathcal{C}^1$  maximal solution to the Cauchy problem

$$\begin{cases} \partial_s \phi(s, t, x) = a(s, \phi(s, t, x)), \\ \phi(t, t, x) = x, \end{cases} \quad (2.4)$$

which is defined on a certain subinterval  $[e(t, x), f(t, x)]$  of  $[0, T]$  (which is closed since  $[0, 1]$  is compact), and with possibly  $e(t, x)$  and/or  $f(t, x) = t$ . Let

$$\text{Dom } \phi = \{(s, t, x); (t, x) \in [0, T] \times [0, 1], s \in [e(t, x), f(t, x)]\}$$

denote the domain of  $\phi$ .

Note that

$$e(t, x) > 0 \Rightarrow \phi(e(t, x), t, x) = 0. \quad (2.5)$$

We take into account the influence of the boundaries by introducing the sets

$$P := \{(s, \phi(s, 0, 0)); s \in [0, f(0, 0)]\},$$

$$I := \{(t, x) \in [0, T] \times [0, 1] \setminus P; e(t, x) = 0\},$$

$$J := \{(t, x) \in [0, T] \times [0, 1] \setminus P; \phi(e(t, x), t, x) = 0\}.$$

(See Figure 1.) Note that both  $I$  and  $J$  are open in  $[0, T] \times [0, 1]$ .

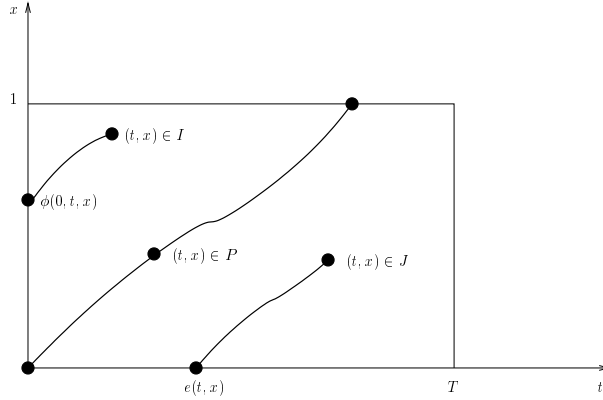


FIGURE 1. Partition of  $[0, T] \times [0, 1]$  into  $I \cup P \cup J$ .

**Proposition 2.1.** *Let  $a$  satisfying (2.2), let  $L = \|a\|_{L^\infty(0,T;\text{Lip}([0,1]))}$ , and let*

$$K := \max(1, \|a\|_{\mathcal{C}^0([0,T] \times [0,1])})e^{LT}.$$

*Then  $\phi$  is  $K$ -Lipschitz on its domain; that is, for all  $(s_1, t_1, x_1), (s_2, t_2, x_2) \in \text{Dom } \phi$*

$$|\phi(s_1, t_1, x_1) - \phi(s_2, t_2, x_2)| \leq K(|s_1 - s_2| + |t_1 - t_2| + |x_1 - x_2|). \quad (2.6)$$

The proof of Proposition 2.1 is given in appendix, for the sake of completeness.

We can now study the regularity of  $e$ .

**Proposition 2.2.** *Let  $a$  be as in Proposition 2.1, let  $(t, x) \in [0, T] \times [0, 1]$ , let  $\{a_n\} \subset \mathcal{C}^0([0, T] \times [0, 1]) \cap L^\infty(0, T; \text{Lip}([0, 1]))$  be a sequence such that  $\|a_n\|_{L^\infty(0, T; \text{Lip}([0, 1]))}$  is bounded and*

$$\|a_n - a\|_{\mathcal{C}^0([0, T] \times [0, 1])} \rightarrow 0 \quad \text{as } n \rightarrow +\infty,$$

*and let  $\{(t_n, x_n)\} \subset [0, T] \times [0, 1]$  be a sequence such that  $(t_n, x_n) \rightarrow (t, x)$ . Then*

$$e_n(t_n, x_n) \rightarrow e(t, x). \quad (2.7)$$

*Proof.* We use again the extension operator  $\Pi$  introduced in the proof of Proposition 2.1 (see the appendix) and set  $\tilde{a}_n = \Pi(a_n)$  and  $\tilde{a} = \Pi(a)$ . Let  $\tilde{\phi}_n$  and  $\tilde{\phi}$  denote their respective flows. Recall that  $\tilde{\phi}$  and  $\phi$  coincide on  $\text{Dom } \phi$  (resp.  $\tilde{\phi}_n$  and  $\phi_n$  coincide on  $\text{Dom } \phi_n$ ). From

$$\begin{aligned} |\partial_s[\tilde{\phi}_n(s, t, x) - \tilde{\phi}(s, t, x)]| &= |\tilde{a}_n(s, \tilde{\phi}_n(s, t, x)) - \tilde{a}(s, \tilde{\phi}(s, t, x))| \\ &\leq |\tilde{a}_n(s, \tilde{\phi}_n(s, t, x)) - \tilde{a}(s, \tilde{\phi}_n(s, t, x))| \\ &\quad + |\tilde{a}(s, \tilde{\phi}_n(s, t, x)) - \tilde{a}(s, \tilde{\phi}(s, t, x))| \\ &\leq \|\tilde{a}_n - \tilde{a}\|_{L^\infty(\mathbb{R}^2)} + \|\tilde{a}\|_{L^\infty(\mathbb{R}; \text{Lip}(\mathbb{R}))} |\tilde{\phi}_n(s, t, x) - \tilde{\phi}(s, t, x)|, \end{aligned}$$

(5.39), (5.40) and Gronwall's lemma, we infer that for all  $n \geq 0$  and all  $(s, t, x) \in [0, T]^2 \times [0, 1]$ , we have

$$|\tilde{\phi}_n - \tilde{\phi}(s, t, x)| \leq T\|a_n - a\|_{\mathcal{C}^0([0, T] \times [0, 1])}e^{T\|a\|_{L^\infty(0, T; \text{Lip}([0, 1])}}. \quad (2.8)$$

It may be seen that

$$e_n(t_n, x_n) = \min\{s \in [0, t_n]; \forall r \in [s, t_n], \tilde{\phi}_n(r, t_n, x_n) \in [0, 1]\}.$$

- If  $(t, x) \in I$ , then since we have excluded the characteristic coming from  $(0, 0)$ , we have that

$$\inf_{s \in [0, t]} \text{dist}((s, \phi(s, t, x)), [0, t] \times \{0\}) > 0,$$

where  $\text{dist}((t, x), F) = \inf_{(t', x') \in F} (|t - t'| + |x - x'|)$ . So we infer from (2.8) that for  $n$  large enough  $\phi_n(\cdot, t, x)$  is defined on  $[0, t]$ , i.e.  $e_n(t, x) = 0$ . Then (2.7) is obvious.

- From now on, we assume that  $(t, x) \in J \cup P$ . We claim that

$$\limsup_{n \rightarrow \infty} e_n(t_n, x_n) \leq e(t, x). \quad (2.9)$$

Indeed, if  $e(t, x) = t$ , then

$$\limsup_{n \rightarrow \infty} e_n(t_n, x_n) \leq \limsup_{n \rightarrow \infty} t_n = t = e(t, x).$$

Otherwise, we have  $e(x, t) < t$ , and using (2.3) we obtain for any  $\epsilon \in (0, (t - e(t, x))/2)$ ,

$$c\epsilon \leq \phi(s, t, x) \leq 1 - c\epsilon, \quad \forall s \in [e(t, x) + \epsilon, t - \epsilon]. \quad (2.10)$$

However, we have for  $n$  large enough

$$\|\tilde{\phi}_n - \tilde{\phi}\|_{\mathcal{C}^0([0, T]^2 \times [0, 1])} \leq \frac{c\epsilon}{4}, \quad (2.11)$$

$$|\tilde{\phi}_n(s, t_n, x_n) - \tilde{\phi}_n(s, t, x)| \leq \frac{c\epsilon}{4}, \quad (2.12)$$

the second estimate coming from the uniform bound on  $\|a_n\|_{L^\infty(0, T; \text{Lip}([0, 1]))}$  and Proposition 2.1. Combining (2.10), (2.11) and (2.12), we see that for  $n$  large and for all  $s \in [e(t, x) + \epsilon, t - \epsilon]$ ,  $\phi_n(s, t_n, x_n)$  is well defined and

$$\phi_n(s, t_n, x_n) \geq \frac{c\epsilon}{2}.$$

This yields  $\limsup_{n \rightarrow \infty} e_n(t_n, x_n) \leq e(t, x) + \epsilon$ , and since  $\epsilon$  was arbitrarily small, (2.9) follows.

If  $(t, x) \in P$  the proof of (2.7) is complete, for  $\liminf_{n \rightarrow \infty} e_n(t_n, x_n) \geq 0 = e(t, x)$ . Assume finally that  $(t, x) \in J$ , so that  $e(t, x) > 0$ . Pick any  $s \in (0, e(t, x))$ . We obviously have  $\tilde{\phi}(s, t, x) < 0$ , thanks to the lower bound on  $\tilde{a}$  (see (5.41)). But we know from (2.8) and Proposition 2.1 that

$$\tilde{\phi}_n(s, t_n, x_n) \xrightarrow{n \rightarrow +\infty} \tilde{\phi}(s, t, x),$$

and hence for  $n$  large enough,  $\tilde{\phi}_n(s, t_n, x_n) < 0$  and  $s < e(t_n, x_n)$ . Thus, we conclude that  $\liminf_{n \rightarrow \infty} e_n(t_n, x_n) \geq s$ . As  $s$  was arbitrarily close to  $e(t, x)$ , we end up with

$$\liminf_{n \rightarrow \infty} e_n(t_n, x_n) \geq e(t, x).$$

The proof of (2.7) is complete.  $\square$

**Remark 1.** (1) For  $a_n = a$ , this shows that  $e$  is continuous on  $[0, T] \times [0, 1]$ .

(2) Since  $[0, T] \times [0, 1]$  is compact, Proposition 2.2 implies that  $e_n$  converges uniformly toward  $e$  on  $[0, T] \times [0, 1]$ .

**Proposition 2.3.** If, in addition to (2.2)-(2.3), we have that  $\partial_x a \in \mathcal{C}^0([0, T] \times [0, 1])$ , then  $\phi$  is  $\mathcal{C}^1$  on  $\text{Dom } \phi$  and  $e$  is  $\mathcal{C}^1$  on  $[0, T] \times [0, 1] \setminus P$ , with for  $(t, x) \in J$

$$\partial_t e(t, x) = \frac{a(t, x) \exp(-\int_{e(t, x)}^t \partial_x a(r, \phi(r, t, x)) dr)}{a(e(t, x), 0)}, \quad \partial_x e(t, x) = -\frac{\exp(-\int_{e(t, x)}^t \partial_x a(r, \phi(r, t, x)) dr)}{a(e(t, x), 0)}. \quad (2.13)$$

*Proof.* The regularity of  $\phi$  is a classical result (see e.g. [21]). If  $(t, x) \in I$ ,  $e(t, x) = 0$  and the result is obvious. For  $(t, x) \in J \cap (0, T) \times (0, 1)$  we have  $\phi(e(t, x), t, x) = 0$  and  $\partial_s \phi(e(t, x), t, x) > 0$ , therefore the Implicit Function Theorem allows us conclude. Finally, for  $(t, x) \in J \setminus (0, T) \times (0, 1)$ , it is sufficient to pass to the limit in (2.13).  $\square$



**Proposition 2.4.** *Let  $a$  fulfill (2.2) and (2.3), and let  $L = \|a\|_{L^\infty(0,T;Lip([0,1]))}$ . Then the function  $e$  is  $\bar{K}$ -Lipschitz on  $[0, T] \times [0, 1]$  where  $\bar{K}$  is given by*

$$\bar{K} = c^{-1} \max \left( 1, \|a\|_{\mathcal{C}^0([0,T] \times [0,1])} \right) e^{LT}.$$

*Proof.* Consider  $(t_1, x_1)$  and  $(t_2, x_2)$  in  $[0, T] \times [0, 1]$ . Let us also suppose that  $e(t_1, x_1) > e(t_2, x_2)$ , the other case being symmetrical. We infer from Proposition 2.1 that

$$|\phi(e(t_1, x_1), t_1, x_1) - \phi(e(t_1, x_1), t_2, x_2)| \leq \max \left( 1, \|a\|_{\mathcal{C}^0([0,T] \times [0,1])} \right) e^{LT} (|t_1 - t_2| + |x_1 - x_2|). \quad (2.14)$$

Since  $e(t_1, x_1) > 0$ , we have that  $\phi(e(t_1, x_1), t_1, x_1) = 0$ , and

$$\phi(e(t_1, x_1), t_2, x_2) \geq c(e(t_1, x_1) - e(t_2, x_2)) \geq 0. \quad (2.15)$$

Therefore we end up with

$$|e(t_1, x_1) - e(t_2, x_2)| \leq \bar{K}(|t_1 - t_2| + |x_1 - x_2|). \quad (2.16)$$

□

**2.4. Strong solutions.** Let  $a \in \mathcal{C}^0([0, T]; \mathcal{C}^1([0, 1]))$ ,  $y_l \in \mathcal{C}^1([0, T])$ , and  $y_0 \in \mathcal{C}^1([0, 1])$  be given, and assume that the following compatibility conditions hold:

$$y_l(0) = y_0(0), \quad y'_l(0) + a(0, 0)y'_0(0) = 0. \quad (2.17)$$

We consider the following boundary initial value problem:

$$\partial_t y + a(t, x)\partial_x y = 0, \quad (t, x) \in (0, T) \times (0, 1), \quad (2.18)$$

$$y(t, 0) = y_l(t), \quad t \in (0, T), \quad (2.19)$$

$$y(0, x) = y_0(x), \quad x \in (0, 1). \quad (2.20)$$

A *strong solution* of (2.18)-(2.20) is any function  $y \in \mathcal{C}^1([0, T] \times [0, 1])$  such that (2.18)-(2.20) hold pointwise.

We define a function  $y : [0, T] \times [0, 1] \rightarrow \mathbb{R}$  in the following way:

$$y(t, x) = \begin{cases} y_l(e(t, x)) & \text{if } (t, x) \in J, \\ y_0(\phi(0, t, x)) & \text{if } (t, x) \in I \cup P. \end{cases} \quad (2.21)$$

**Proposition 2.5.** *Let  $y$  be as in (2.21). Then  $y$  is a strong solution of (2.18)-(2.20). Besides, we have the estimates*

$$\|y\|_{\mathcal{C}^0([0,T] \times [0,1])} \leq \max \left( \|y_0\|_{\mathcal{C}^0([0,1])}, \|y_l\|_{\mathcal{C}^0([0,T])} \right), \quad (2.22)$$

$$\|\nabla y\|_{\mathcal{C}^0([0,T] \times [0,1])} \leq \max \left( \|y'_0\|_{\mathcal{C}^0([0,1])}, \|y'_l\|_{\mathcal{C}^0([0,T])} \frac{\|a\|_{\mathcal{C}^0([0,T] \times [0,1])}}{c} \right) \exp \left( T \|\partial_x a\|_{\mathcal{C}^0([0,T] \times [0,1])} \right). \quad (2.23)$$

*Proof.* One can see that  $y$  is of class  $\mathcal{C}^1$  on  $I$  and  $J$ , with the derivatives given by:

$$\begin{aligned}\partial_t y(t, x) &= y'_l(e(t, x)) \frac{a(t, x)}{a(e(t, x), 0)} \exp \left( - \int_{e(t, x)}^t \partial_x a(s, \phi(s, t, x)) ds \right), \quad \forall (t, x) \in J, \\ \partial_x y(t, x) &= -y'_l(e(t, x)) \frac{1}{a(e(t, x), 0)} \exp \left( - \int_{e(t, x)}^t \partial_x a(s, \phi(s, t, x)) ds \right), \quad \forall (t, x) \in J, \\ \partial_t y(t, x) &= -y'_0(\phi(0, t, x)) a(t, x) \exp \left( - \int_0^t \partial_x a(s, \phi(s, t, x)) ds \right), \quad \forall (t, x) \in I, \\ \partial_x y(t, x) &= y'_0(\phi(0, t, x)) \exp \left( - \int_0^t \partial_x a(s, \phi(s, t, x)) ds \right), \quad \forall (t, x) \in I.\end{aligned}$$

It follows from the first equation in (2.17) and the continuity of  $e$  that  $y$  is continuous at each point of  $P$ . Note that  $y$  is differentiable in directions  $t$  and  $x$  in the following way: for all  $t \in (0, f(0, 0))$

$$\begin{aligned}\lim_{h \rightarrow 0^+} \frac{y(t+h, \phi(t, 0, 0))}{h} &= y'_l(0) \frac{a(t, \phi(t, 0, 0))}{a(0, 0)} \exp \left( - \int_0^t \partial_x a(s, \phi(s, 0, 0)) ds \right), \\ \lim_{h \rightarrow 0^-} \frac{y(t+h, \phi(t, 0, 0))}{h} &= -y'_0(0) a(t, \phi(t, 0, 0)) \exp \left( - \int_0^t \partial_x a(s, \phi(s, 0, 0)) ds \right), \\ \lim_{h \rightarrow 0^+} \frac{y(t, \phi(t, 0, 0) + h)}{h} &= y'_0(0) \exp \left( - \int_0^t \partial_x a(s, \phi(s, 0, 0)) ds \right), \\ \lim_{h \rightarrow 0^-} \frac{y(t, \phi(t, 0, 0) + h)}{h} &= -y'_l(0) \frac{1}{a(0, 0)} \exp \left( - \int_0^t \partial_x a(s, \phi(s, 0, 0)) ds \right).\end{aligned}$$

Using the second equation in (2.17), we see that  $y \in \mathcal{C}^1([0, T] \times [0, 1])$ . The fact that  $y$  satisfies (2.1) follows from a straightforward calculation.  $\square$

**2.5. Weak solutions.** Now we consider the case when  $a \in \mathcal{C}^0([0, T] \times [0, 1])$  and  $\partial_x a \in L^\infty((0, T) \times (0, 1))$ . We still assume that

$$a(t, x) \geq c > 0, \quad \forall (t, x) \in [0, T] \times [0, 1]. \quad (2.24)$$

We begin by introducing the space:

$$\mathcal{T} = \{\psi \in \mathcal{C}^1([0, T] \times [0, 1]); \psi(t, 1) = \psi(T, x) = 0 \quad \forall (t, x) \in [0, T] \times [0, 1]\}. \quad (2.25)$$

We say that a function  $y \in L^1((0, T) \times (0, 1))$  is a *weak solution* of (2.18)-(2.20) if for any  $\psi \in \mathcal{T}$  we have

$$\begin{aligned}\iint_{(0, T) \times (0, 1)} y(t, x) (\psi_t(t, x) + a(t, x) \psi_x(t, x) + a_x(t, x) \psi(t, x)) dt dx \\ + \int_0^T \psi(t, 0) y_l(t) a(t, 0) dt + \int_0^1 \psi(0, x) y_0(x) dx = 0.\end{aligned} \quad (2.26)$$

Using the results of Section 2.4, it is clear that a strong solution is also a weak solution. Conversely, any weak solution which is in  $C^1([0, T] \times [0, 1])$  is a strong solution. Note that the definition of weak solution makes sense for  $y_l \in L^1(0, T)$  and  $y_0 \in L^1(0, 1)$ .

**Proposition 2.6.** *Let us suppose that  $a$ ,  $y_l$  and  $y_0$  are uniformly Lipschitz continuous with Lipschitz constants  $L$ ,  $L_l$  and  $L_0$ , respectively, and that  $y_l(0) = y_0(0)$ . Then the function  $y$  defined by*

$$y(t, x) = \begin{cases} y_l(e(t, x)) & \text{if } (t, x) \in J, \\ y_0(\phi(0, t, x)) & \text{if } (t, x) \in I \cup P, \end{cases} \quad (2.27)$$

is a weak solution of (2.18)-(2.20). Furthermore,  $y$  is  $M$ -Lipschitz continuous on  $[0, T] \times [0, 1]$  with  $M$  defined by

$$M := \max\left(\frac{L_l}{c}, L_0\right) \max\left(1, \|a\|_{\mathcal{C}^0([0, T] \times [0, 1])}\right) e^{LT}. \quad (2.28)$$

Finally,  $y$  is the unique solution in the class  $\text{Lip}([0, T] \times [0, 1])$  of system (2.18)-(2.20), with (2.18) understood in the distributional sense, and (2.19)-(2.20) pointwise.

*Proof.* Using standard regularization arguments, it is possible to find  $a^n \in \mathcal{C}^1([0, T] \times [0, 1])$ ,  $y_0^n \in \mathcal{C}^1([0, 1])$  and  $y_l^n \in \mathcal{C}^1([0, 1])$  such that:

$$\forall \epsilon \in (0, \min(T, 1)) \quad \|a^n - a\|_{\mathcal{C}^0([0, T] \times [0, 1])} + \|y_0^n - y_0\|_{\mathcal{C}^0([0, 1])} + \|y_l^n - y_l\|_{\mathcal{C}^0([0, T])} \xrightarrow{n \rightarrow +\infty} 0, \quad (2.29)$$

$$\|a_n\|_{\mathcal{C}^0([0, T] \times [0, 1])} \leq 2\|a\|_{\mathcal{C}^0([0, T] \times [0, 1])}, \quad \|\partial_x a_n\|_{L^\infty((0, T) \times (0, 1))} \leq 2\|\partial_x a\|_{L^\infty((0, T) \times (0, 1))}, \quad (2.30)$$

$$\|y_0^n\|_{\mathcal{C}^0([0, 1])} \leq 2\|y_0\|_{\mathcal{C}^0([0, 1])}, \quad \|y_0^{n'}\|_{L^\infty(0, 1)} \leq 2\|y_0'\|_{L^\infty(0, 1)}, \quad (2.31)$$

$$\|y_l^n\|_{\mathcal{C}^0([0, T])} \leq 2\|y_l\|_{\mathcal{C}^0([0, T])}, \quad \|y_l^{n'}\|_{L^\infty(0, T)} \leq 2\|y_l'\|_{L^\infty(0, T)}, \quad (2.32)$$

$$y_0^n(0) = y_l^n(0), \quad y_0^{n'}(0) = y_l^{n'}(0) = 0 \quad \forall n \in \mathbb{N}. \quad (2.33)$$

Using Proposition 2.5 we infer the existence of a strong solution  $y^n \in \mathcal{C}^1([0, T] \times [0, 1])$  of

$$\begin{cases} \partial_t y^n + a^n \partial_x y^n = 0, \\ y^n(t, 0) = y_l^n(t), \quad y^n(0, x) = y_0^n(x), \end{cases} \quad \forall (t, x) \in (0, T) \times (0, 1). \quad (2.34)$$

$y^n$  is given by (2.21), with  $y_l, y_0, e$  and  $\phi$  replaced by  $y_l^n, y_0^n, e^n$  and  $\phi^n$ , respectively. Note that  $(t, x) \in I^n$  (resp.  $(t, x) \in J^n$ ) for  $n$  large enough if  $(t, x) \in I$  (resp.  $(t, x) \in J$ ). Using Proposition 2.2, (2.8) and (2.21), we see that

$$y^n(t, x) \xrightarrow{n \rightarrow +\infty} y(t, x), \quad \forall (t, x) \in I \cup J. \quad (2.35)$$

Note that  $P = [0, T] \times [0, 1] \setminus (I \cup J)$  has zero Lebesgue measure. An application of the dominated convergence theorem yields

$$\|y^n - y\|_{L^1((0, T) \times (0, 1))} \xrightarrow{n \rightarrow +\infty} 0.$$

Using the other convergence assumptions about  $a^n$ ,  $y_l^n$ , and  $y_0^n$ , we can pass to the limit in (2.26). This shows that  $y$  is a weak solution of (2.1).

To prove the regularity of  $y$  we distinguish two cases.

Assume first that both  $(t_1, x_1)$  and  $(t_2, x_2)$  are in  $J \cup P$ . Using (2.21) and Proposition 2.4, we

have that

$$\begin{aligned} |y(t_1, x_1) - y(t_2, x_2)| &= |y_l(e(t_1, x_1)) - y_l(e(t_2, x_2))| \\ &\leq L_l |e(t_1, x_1) - e(t_2, x_2)| \\ &\leq \frac{L_l}{c} \max \left( 1, \|a\|_{\mathcal{C}^0([0,T] \times [0,1])} \right) e^{LT} (|t_1 - t_2| + |x_1 - x_2|). \end{aligned}$$

Next, if we assume that  $(t_1, x_1)$  and  $(t_2, x_2)$  are in  $I \cup P$ , then we can use (2.21) and Proposition 2.1 to obtain that

$$\begin{aligned} |y(t_1, x_1) - y(t_2, x_2)| &= |y_0(\phi(0, t_1, x_1)) - y_0(\phi(0, t_2, x_2))| \\ &\leq L_0 |\phi(0, t_1, x_1) - \phi(0, t_2, x_2)| \\ &\leq L_0 \max \left( 1, \|a\|_{\mathcal{C}^0([0,T] \times [0,1])} \right) e^{LT} (|t_1 - t_2| + |x_1 - x_2|). \end{aligned}$$

Finally, if  $(t_1, x_1) \in J$  and  $(t_2, x_2) \in I$ , we consider an intermediate point on  $P$  belonging to the boundary of the rectangle  $[\min(t_1, t_2), \max(t_1, t_2)] \times [\min(x_1, x_2), \max(x_1, x_2)]$  and use the estimates above.

Let us now check that  $y$  is the only solution to (2.18)-(2.20) in the class  $\text{Lip}([0, T] \times [0, 1])$ . First, picking any  $\psi \in \mathcal{C}_0^\infty((0, T) \times (0, 1))$  in (2.26), we see that (2.18) holds in  $\mathcal{D}'((0, T) \times (0, 1))$ . Note that each term in (2.18) belongs to  $L^\infty((0, T) \times (0, 1))$ , so that (2.18) holds also pointwise a.e. Scaling in (2.18) by  $\psi \in \mathcal{T}$  and comparing to (2.26), we obtain that (2.19) and (2.20) hold a.e., and also everywhere by continuity of  $y_l$ ,  $y_0$ , and  $y$ . Thus  $y$  solves (2.18)-(2.20). If  $\tilde{y} \in \text{Lip}([0, T] \times [0, 1])$  is another solution of (2.18)-(2.20), then  $\hat{y} := y - \tilde{y} \in \text{Lip}([0, T] \times [0, 1])$  solves

$$\partial_t \hat{y} + a(t, x) \partial_x \hat{y} = 0 \quad \text{in } \mathcal{D}'((0, T) \times (0, 1)), \quad (2.36)$$

$$\hat{y}(t, 0) = 0 \quad \text{in } (0, T), \quad (2.37)$$

$$\hat{y}(0, x) = 0 \quad \text{in } (0, 1). \quad (2.38)$$

Scaling in (2.36) by  $2\hat{y}$ , integrating by parts and using (2.3), (2.37), and (2.38), we obtain

$$\|\hat{y}(t)\|_2^2 = \int_0^t \int_0^1 (\partial_x a) |\hat{y}|^2 dx ds - \int_0^t a(t, 1) |\hat{y}(t, 1)|^2 dt \leq L \int_0^t \|\hat{y}(s)\|_2^2 ds.$$

This yields  $\hat{y} \equiv 0$  by Gronwall's lemma. The proof of Proposition 2.6 is complete.  $\square$

### 3. FINITE-TIME BOUNDARY STABILIZATION OF A SYSTEM OF TWO CONSERVATION LAWS

In this section, we consider the system

$$\begin{cases} \partial_t u + \lambda(u, v) \partial_x u = 0, \\ \partial_t v + \mu(u, v) \partial_x v = 0, \end{cases} \quad (t, x) \in (0, +\infty) \times (0, 1). \quad (3.1)$$

where  $\lambda$  and  $\mu$  are given functions with

$$\lambda, \mu \in \mathcal{C}^\infty(\mathbb{R}^2, \mathbb{R}), \quad (3.2)$$

$$\mu(u, v) \leq -c < 0 < c \leq \lambda(u, v), \quad \forall (u, v) \in \mathbb{R}^2 \quad (3.3)$$

for some constant  $c > 0$ . We aim to prescribe a control in a feedback form on the boundary conditions  $u(t, 0)$  and  $v(t, 1)$  so that for some time  $T$  we have for any small (in  $\text{Lip}([0, 1])$ ) initial data  $u_0$  and  $v_0$

$$u(T, x) = v(T, x) = 0, \quad \forall x \in (0, 1). \quad (3.4)$$

**Remark 2.** (1) *If we intend to stabilize the system around a non null (but constant) equilibrium state  $(\bar{u}, \bar{v}) \in \mathbb{R}^2$ , it is sufficient to consider the new unknowns  $\tilde{u} := u - \bar{u}$ ,  $\tilde{v} := v - \bar{v}$  that satisfy a system similar to (3.1), and to stabilize  $(\tilde{u}, \tilde{v})$  around  $(0, 0)$ .*  
 (2) *Note that, since we are only interested in proving a local stabilization result, the condition (3.3) is not too much restrictive. It should be seen as  $\lambda(\bar{u}, \bar{v}) > 0$  and  $\mu(\bar{u}, \bar{v}) < 0$ .*

After introducing the boundary feedback law, we will show the existence and uniqueness of the solution to the closed loop system and check that the property (3.4) indeed holds for this choice of feedback law.

We now come back to the quasilinear system (3.1) that we complete as follows:

$$\begin{cases} \partial_t u + \lambda(u, v) \partial_x u = 0, & (t, x) \in (0, +\infty) \times (0, 1), \\ \partial_t v + \mu(u, v) \partial_x v = 0, & (t, x) \in (0, +\infty) \times (0, 1), \end{cases} \quad (3.5)$$

$$\begin{cases} \frac{d}{dt} u(t, 0) = -K \text{sgn}(u(t, 0)) |u(t, 0)|^\gamma, & t > 0, \\ \frac{d}{dt} v(t, 1) = -K \text{sgn}(v(t, 1)) |v(t, 1)|^\gamma, & t > 0, \end{cases} \quad (3.6)$$

$$u(0, x) = u_0(x), \quad v(0, x) = v_0(x), \quad x \in (0, 1) \quad (3.7)$$

with  $(K, \gamma) \in (0, +\infty) \times (0, 1)$  arbitrarily chosen. We aim to use Schauder fixed-point theorem to prove the local in time existence of solutions  $(u, v)$  of (3.5)-(3.7) in some class of Lipschitz continuous functions. By *solution*, we mean that (3.5) is satisfied in the distributional sense, and that (3.6)-(3.7) are satisfied pointwise. Actually, we shall use the results of the previous section and define  $u$  as the weak solution of the transport equation (2.18)-(2.20) with  $a(t, x) = \lambda(\tilde{u}(t, x), \tilde{v}(t, x))$  for some given pair  $(\tilde{u}, \tilde{v})$  in the same class,  $y_l(t) = u_l(t)$  (see below (3.10)), and  $y_0(x) = u_0(x)$ , and similarly for  $v$ .

**3.1. Notations.** Let  $C_1 > 0$  and  $C_2 > 0$  be given, and pick any  $u_0, v_0 \in \text{Lip}([0, 1])$  with

$$\max(\|u_0\|_\infty, \|v_0\|_\infty) \leq C_1, \quad (3.8)$$

$$\max(\|u'_0\|_\infty, \|v'_0\|_\infty) \leq C_2. \quad (3.9)$$

Let

$$T := \frac{1}{c} + \frac{C_1^{1-\gamma}}{(1-\gamma)K}.$$

We define  $u_l$  and  $v_r$  as the solutions of the following ODEs

$$\begin{cases} \frac{d}{dt} u_l(t) = -K \text{sgn}(u_l(t)) |u_l(t)|^\gamma, & u_l(0) = u_0(0), \\ \frac{d}{dt} v_r(t) = -K \text{sgn}(v_r(t)) |v_r(t)|^\gamma, & v_r(0) = v_0(1). \end{cases} \quad (3.10)$$

An obvious calculation gives

$$u_l(t) = \begin{cases} \operatorname{sgn}(u_0(0)) (|u_0(0)|^{1-\gamma} - (1-\gamma)Kt)^{\frac{1}{1-\gamma}} & \text{if } 0 \leq t \leq \frac{|u_0(0)|^{1-\gamma}}{(1-\gamma)K}, \\ 0 & \text{if } t \geq \frac{|u_0(0)|^{1-\gamma}}{(1-\gamma)K}, \end{cases}$$

and

$$v_r(t) = \begin{cases} \operatorname{sgn}(v_0(1)) (|v_0(1)|^{1-\gamma} - (1-\gamma)Kt)^{\frac{1}{1-\gamma}} & \text{if } 0 \leq t \leq \frac{|v_0(1)|^{1-\gamma}}{(1-\gamma)K}, \\ 0 & \text{if } t \geq \frac{|v_0(1)|^{1-\gamma}}{(1-\gamma)K}, \end{cases}$$

Clearly

$$\forall t \geq T - \frac{1}{c}, \quad v_r(t) = u_l(t) = 0, \quad (3.11)$$

$$\max(\|u_l\|_\infty, \|v_r\|_\infty) \leq C_1, \quad (3.12)$$

$$\max(\|u'_l\|_\infty, \|v'_r\|_\infty) \leq KC_1^\gamma. \quad (3.13)$$

Let us also introduce

$$M_1 := \max \left( \|\lambda\|_{\mathcal{C}^0([-C_1, C_1]^2)}, \|\mu\|_{\mathcal{C}^0([-C_1, C_1]^2)} \right), \quad (3.14)$$

$$M_2 := \max \left( \|\partial_u \mu\|_{\mathcal{C}^0([-C_1, C_1]^2)}, \|\partial_v \mu\|_{\mathcal{C}^0([-C_1, C_1]^2)}, \|\partial_u \lambda\|_{\mathcal{C}^0([-C_1, C_1]^2)}, \|\partial_v \lambda\|_{\mathcal{C}^0([-C_1, C_1]^2)} \right). \quad (3.15)$$

Let us pick a positive number  $C_3$ . Let  $\mathcal{D}$  denote the domain

$$\mathcal{D} := \left\{ (u, v) \in \operatorname{Lip}([0, T] \times [0, 1])^2; \max(\|u\|_\infty, \|v\|_\infty) \leq C_1, \text{ and } u \text{ and } v \text{ are } C_3\text{-Lipschitz} \right\}. \quad (3.16)$$

Let us equip the domain  $\mathcal{D}$  with the topology of the uniform convergence. Then, by Ascoli-Arzelà theorem,  $\mathcal{D}$  is a compact set in  $\mathcal{C}^0([0, T] \times [0, 1])^2$ .

The main result in this section is the following

**Theorem 1.** *Assume that  $C_1 > 0$  and  $C_2 > 0$  are such that*

$$TM_2 \max(1, M_1) \max\left(\frac{KC_1^\gamma}{c}, C_2\right) \leq \frac{1}{2e} \quad (3.17)$$

and let  $C_3 = (2TM_2)^{-1}$ . Pick any pair  $(u_0, v_0) \in \operatorname{Lip}([0, 1])^2$  satisfying (3.8)-(3.9). Then there exists a unique solution  $(u, v)$  of (3.5)-(3.7) in the class  $\mathcal{D}$ . Furthermore, the solution is global in time with  $u(t, \cdot) = v(t, \cdot) = 0$  for  $t \geq T$ . Finally, the equilibrium state  $(0, 0)$  is stable in  $\operatorname{Lip}([0, 1])^2$  for (3.5)-(3.7); that is

$$\|(u, v)\|_{L^\infty(\mathbb{R}^+; \operatorname{Lip}([0, 1])^2)} \rightarrow 0 \quad \text{as} \quad \|(u_0, v_0)\|_{\operatorname{Lip}([0, 1])^2} \rightarrow 0. \quad (3.18)$$

The first task consists in constructing a solution of the closed loop system as a fixed point of a certain operator.

**3.2. Definition of the operator.** If  $(\tilde{u}, \tilde{v}) \in \mathcal{D}$  are given, we define  $(u, v) = \mathcal{F}(\tilde{u}, \tilde{v})$  as follows: the function  $u$  is the weak solution of the system

$$\begin{cases} \partial_t u + \lambda(\tilde{u}, \tilde{v}) \partial_x u = 0, \\ u(t, 0) = u_l(t), \quad u(0, x) = u_0(x), \end{cases} \quad \forall (t, x) \in [0, T] \times [0, 1], \quad (3.19)$$

and the function  $v$  is the weak solution of the system

$$\begin{cases} \partial_t v + \mu(\tilde{u}, \tilde{v}) \partial_x v = 0, \\ v(t, 1) = v_r(t), \quad v(0, x) = v_0(x), \end{cases} \quad \forall (t, x) \in [0, T] \times [0, 1]. \quad (3.20)$$

**3.3. Stability of the domain.** In this part, we show that for a certain choice of  $C_1, C_2, C_3$ , we have

$$\mathcal{F}(\mathcal{D}) \subset \mathcal{D}.$$

We first apply the results of Section 2 to get the following

**Lemma 1.** *Let  $C_1, C_2, C_3$  be any positive numbers, and let  $u_0, v_0 \in \text{Lip}([0, 1])$  satisfying (3.8)-(3.9). For given  $(\tilde{u}, \tilde{v}) \in \mathcal{D}$ , let  $(u, v) = \mathcal{F}(\tilde{u}, \tilde{v})$ . Then the functions  $u$  and  $v$  are Lipschitz continuous on  $[0, T] \times [0, 1]$  and they satisfy the following estimates*

$$\max(\|u\|_{\mathcal{C}^0([0, T] \times [0, 1])}, \|v\|_{\mathcal{C}^0([0, T] \times [0, 1])}) \leq C_1, \quad (3.21)$$

$$\begin{aligned} \max(\|\partial_x u\|_{L^\infty((0, T) \times (0, 1))}, \|\partial_x v\|_{L^\infty((0, T) \times (0, 1))}, \|\partial_t u\|_{L^\infty((0, T) \times (0, 1))}, \|\partial_t v\|_{L^\infty((0, T) \times (0, 1))}) \\ \leq \max\left(\frac{KC_1^\gamma}{c}, C_2\right) \max(1, M_1) \exp(2TM_2C_3). \end{aligned} \quad (3.22)$$

*Proof.* Estimate (3.21) follows directly from (3.19), (3.20), (2.27), (3.8) and (3.12).

Estimate (3.22) can be deduced applying (2.28) for (3.19) and (3.20), and using (3.9), (3.13), (3.14), (3.15) and (3.16).  $\square$

Thanks to Lemma 1, we see that the domain  $\mathcal{D}$  is stable by  $\mathcal{F}$  as soon as

$$\max\left(\frac{KC_1^\gamma}{c}, C_2\right) \max(1, M_1) \exp(2TM_2C_3) \leq C_3. \quad (3.23)$$

This can be written as

$$\max\left(\frac{KC_1^\gamma}{c}, C_2\right) \leq \frac{C_3 \exp(-2TM_2C_3)}{\max(1, M_1)}. \quad (3.24)$$

For given  $C_1$  and  $C_2$ ,  $T, M_1$  and  $M_2$  are fixed. Note that  $T, M_1$  and  $M_2$  are independent of  $C_2$ , and that they are nondecreasing in  $C_1$ . Therefore, as a function of  $C_3$  the supremum of the right-hand side of (3.24) is attained for  $C_3 = (2TM_2)^{-1}$ , and for this value of  $C_3$  the condition on  $C_1$  and  $C_2$  for the domain to be stable reads

$$TM_2 \max(1, M_1) \max\left(\frac{KC_1^\gamma}{c}, C_2\right) \leq \frac{1}{2e}. \quad (3.25)$$

But the term in the left-hand side of (3.25) tends to 0 when  $C_1$  and  $C_2$  tend to 0, so that for  $C_1, C_2$  small enough the condition (3.25) is satisfied and  $\mathcal{D}$  is stable by  $\mathcal{F}$ .

**3.4. Continuity of the operator.** In this part we consider a sequence  $\{(\tilde{u}_n, \tilde{v}_n)\} \subset \mathcal{D}$  and a couple  $(\tilde{u}, \tilde{v}) \in \mathcal{D}$  such that

$$\max \left( \|\tilde{u}_n - \tilde{u}\|_{\mathcal{C}^0([0,T] \times [0,1])}, \|\tilde{v}_n - \tilde{v}\|_{\mathcal{C}^0([0,T] \times [0,1])} \right) \xrightarrow{n \rightarrow +\infty} 0. \quad (3.26)$$

Let us now define

$$(u_n, v_n) = \mathcal{F}(\tilde{u}_n, \tilde{v}_n) \quad \text{for } n \geq 0, \quad \text{and} \quad (u, v) = \mathcal{F}(\tilde{u}, \tilde{v}). \quad (3.27)$$

Our goal in this subsection is to show that

$$\max \left( \|u_n - u\|_{\mathcal{C}^0([0,T] \times [0,1])}, \|v_n - v\|_{\mathcal{C}^0([0,T] \times [0,1])} \right) \xrightarrow{n \rightarrow +\infty} 0. \quad (3.28)$$

We need the following

**Lemma 2.** *For almost all  $(t, x) \in [0, T] \times [0, 1]$ , we have*

$$(u_n(t, x), v_n(t, x)) \xrightarrow{n \rightarrow +\infty} (u(t, x), v(t, x)). \quad (3.29)$$

*Proof.* Let us show that  $u_n(t, x) \rightarrow u(t, x)$ , the convergence  $v_n(t, x) \rightarrow v(t, x)$  being similar.

The fact that  $(\tilde{u}_n, \tilde{v}_n)$  converges uniformly toward  $(\tilde{u}, \tilde{v})$  on  $[0, T] \times [0, 1]$  implies that  $\lambda(\tilde{u}_n, \tilde{v}_n)$  converges uniformly toward  $\lambda(\tilde{u}, \tilde{v})$  on  $[0, T] \times [0, 1]$ . Furthermore, since  $(\tilde{u}_n, \tilde{v}_n) \in \mathcal{D}$  for all  $n$ , we see that the functions  $\lambda(\tilde{u}_n, \tilde{v}_n)$  are uniformly Lipschitz continuous for  $n \geq 0$ . This will allow us to use Proposition 2.2. To this end, we consider the flow  $\phi_n$  (resp.  $\phi$ ) of  $\lambda(\tilde{u}_n, \tilde{v}_n)$  (resp.  $\lambda(\tilde{u}, \tilde{v})$ ). In the same way, we define  $e_n$  and  $e$ ,  $I_n$  and  $I$ ,  $J_n$  and  $J$ ,  $P_n$  and  $P$ . Using (2.27) we have that

$$u_n(t, x) = \begin{cases} u_l(e_n(t, x)) & \text{if } (t, x) \in J_n, \\ u_0(\phi_n(0, t, x)) & \text{if } (t, x) \in I_n \cup P_n, \end{cases} \quad (3.30)$$

and also

$$u(t, x) = \begin{cases} u_l(e(t, x)) & \text{if } (t, x) \in J, \\ u_0(\phi(0, t, x)) & \text{if } (t, x) \in I \cup P. \end{cases} \quad (3.31)$$

We infer from Proposition 2.2 that

$$e_n(t, x) \xrightarrow{n \rightarrow +\infty} e(t, x), \quad \forall (t, x) \in [0, T] \times [0, 1]. \quad (3.32)$$

This shows in particular that if  $(t, x) \in J$ , then  $e(t, x) > 0$  and hence  $e_n(t, x) > 0$  for  $n$  large enough, i.e.  $(t, x) \in J_n$  for  $n$  large enough. Therefore

$$u_n(t, x) \xrightarrow{n \rightarrow +\infty} u(t, x), \quad \forall (t, x) \in J.$$

Now if  $(t, x) \in I$ , then  $e(t, x) = 0$  and  $\phi(0, t, x) > 0$ . Since  $\lambda \geq c > 0$ , this implies the existence of  $\epsilon > 0$  such that

$$\epsilon < \phi(s, t, x), \quad \forall s \in [0, t]. \quad (3.33)$$

Combined with (2.8), this shows that for  $n$  large enough  $e_n(t, x) = 0$  and  $\phi_n(0, t, x) \rightarrow \phi(0, t, x)$ , so we conclude that

$$u_n(t, x) = u_0(\phi_n(0, t, x)) \xrightarrow{n \rightarrow +\infty} u_0(\phi(0, t, x)) = u(t, x). \quad (3.34)$$

Finally,  $P$  is clearly negligible and  $I \cup P \cup L = [0, T] \times [0, 1]$ .  $\square$



To strengthen this convergence, we just need to recall that for every  $n \geq 0$ , we have  $(u_n, v_n) \in \mathcal{D}$  which is compact in  $\mathcal{C}^0([0, T] \times [0, 1])$ . According to Lemma 2, the only possible limit point is  $(u, v)$  and therefore we get the convergence of the whole sequence in  $\mathcal{D}$ ; that is,

$$\max \left( \|u_n - u\|_{\mathcal{C}^0([0, T] \times [0, 1])}, \|v_n - v\|_{\mathcal{C}^0([0, T] \times [0, 1])} \right) \xrightarrow{n \rightarrow +\infty} 0. \quad (3.35)$$

This shows that the operator  $\mathcal{F}$  is continuous on the domain  $\mathcal{D}$ , which is a convex compact set in  $\mathcal{C}^0([0, T] \times [0, 1])^2$ . It follows then from Schauder fixed-point theorem that  $\mathcal{F}$  has a fixed-point. This proves the existence of solutions on the time interval  $[0, T]$ .

**3.5. Uniqueness of the solution.** Let  $u_0, v_0 \in \text{Lip}([0, 1])$  be as in (3.8)-(3.9). Assume given two pairs  $(u^1, v^1), (u^2, v^2) \in \mathcal{D}$  of solutions of (3.5)-(3.7); that is, if  $u_l$  and  $v_r$  are defined as in (3.10), then  $u^i, i = 1, 2$ , is a (weak) solution of

$$\begin{cases} \partial_t u^i + \lambda(u^i, v^i) \partial_x u^i = 0, \\ u^i(t, 0) = u_l(t), \quad u^i(0, x) = u_0(x), \end{cases} \quad (t, x) \in (0, T) \times (0, 1),$$

while  $v^i, i = 1, 2$ , is a (weak) solution of

$$\begin{cases} \partial_t v^i + \mu(u^i, v^i) \partial_x v^i = 0, \\ v^i(t, 1) = v_r(t), \quad v^i(0, x) = v_0(x), \end{cases} \quad (t, x) \in (0, T) \times (0, 1).$$

Let  $\hat{u} = u^1 - u^2$  and  $\hat{v} = v^1 - v^2$ . Note that  $\hat{u}, \hat{v} \in \text{Lip}([0, T] \times [0, 1]) = W^{1,\infty}((0, T) \times (0, 1))$  and that  $\hat{u}, \hat{v}$  fulfill

$$\partial_t \hat{u} + \lambda^1 \partial_x \hat{u} + \hat{\lambda} \partial_x u^2 = 0, \quad (t, x) \in (0, T) \times (0, 1), \quad (3.36)$$

$$\partial_t \hat{v} + \mu^1 \partial_x \hat{v} + \hat{\mu} \partial_x v^2 = 0, \quad (t, x) \in (0, T) \times (0, 1), \quad (3.37)$$

$$\hat{u}(t, 0) = \hat{v}(t, 1) = 0, \quad \hat{u}(0, x) = \hat{v}(0, x) = 0, \quad (3.38)$$

where  $\lambda^i = \lambda(u^i, v^i)$ ,  $\mu^i = \mu(u^i, v^i)$ , and  $\hat{\lambda} = \lambda^1 - \lambda^2$ ,  $\hat{\mu} = \mu^1 - \mu^2$ .

Multiplying in (3.36) by  $2\hat{u}$ , in (3.37) by  $2\hat{v}$ , integrating over  $(0, t) \times (0, 1)$ , and adding the two equations gives

$$\|\hat{u}(t)\|_2^2 + \|\hat{v}(t)\|_2^2 + 2 \int_0^t \int_0^1 (\lambda^1 \hat{u} \partial_x \hat{u} + \mu^1 \hat{v} \partial_x \hat{v}) dx ds + 2 \int_0^t \int_0^1 (\hat{\lambda} \hat{u} \partial_x u^2 + \hat{\mu} \hat{v} \partial_x v^2) dx ds = 0.$$

Using (3.38) and an integration by parts, we obtain

$$\begin{aligned} & 2 \int_0^t \int_0^1 (\lambda^1 \hat{u} \partial_x \hat{u} + \mu^1 \hat{v} \partial_x \hat{v}) \\ &= - \int_0^t \int_0^1 [(\partial_x \lambda^1) |\hat{u}|^2 + (\partial_x \mu^1) |\hat{v}|^2] dx ds + \int_0^t [\lambda^1 |\hat{u}(s, 1)|^2 - \mu^1 |\hat{v}(s, 0)|^2] ds \\ &\geq - \int_0^t \int_0^1 [(\partial_x \lambda^1) |\hat{u}|^2 + (\partial_x \mu^1) |\hat{v}|^2] dx ds \end{aligned}$$

where we used (3.3). On the other hand, since  $\lambda$  and  $\mu$  are  $M_2$ -Lipschitz continuous on  $[-C_1, C_1]^2$ , we infer that  $\lambda^i$  and  $\mu^i$  are  $2M_2C_3$ -Lipschitz continuous on  $[0, T] \times [0, 1]$ . In particular,

$$\|\partial_x \lambda^1\|_\infty \leq 2M_2C_3, \quad \|\partial_x \mu^1\|_\infty \leq 2M_2C_3$$

and

$$\begin{aligned} |\hat{\lambda}| &\leq M_2(|\hat{u}| + |\hat{v}|), \\ |\hat{\mu}| &\leq M_2(|\hat{u}| + |\hat{v}|). \end{aligned}$$

This yields

$$|2 \int_0^t \int_0^1 (\hat{\lambda} \hat{u} \partial_x u^2 + \hat{\mu} \hat{v} \partial_x v^2) dx ds| \leq 2M_2 C_3 \int_0^t \int_0^1 (|\hat{u}| + |\hat{v}|)^2 dx ds.$$

We conclude that for all  $t \in (0, T)$

$$\|\hat{u}(t)\|_2^2 + \|\hat{v}(t)\|_2^2 \leq 6M_2 C_3 \int_0^t (\|\hat{u}\|_2^2 + \|\hat{v}\|_2^2) ds.$$

This yields  $\hat{u} = \hat{v} \equiv 0$ , by Gronwall's lemma.

**3.6. Finite-time extinction of the maximal solutions.** In this section,  $(u, v)$  denotes the only solution of (3.5)-(3.7) in the class  $\mathcal{D}$ .

**Lemma 3.** *At time  $t = T$  we have*

$$u(T, x) = v(T, x) = 0, \quad \forall x \in [0, 1]. \quad (3.39)$$

*Proof of Lemma 3:* We infer from (3.6) that

$$u(t, 0) = v(t, 1) = 0, \quad \forall t \geq T - \frac{1}{c}. \quad (3.40)$$

Thanks to (3.2)-(3.3), we have that

$$\lambda(u(t, x), v(t, x)) \geq c > 0 > -c > \mu(u(t, x), v(t, x)), \quad \forall (t, x) \in [0, T] \times [0, 1].$$

Let  $\phi^\lambda$  (resp.  $\phi^\mu$ ) denote the flow of  $\lambda(u, v)$  (resp.  $\mu(u, v)$ ), and let  $e^\lambda$  (resp.  $e^\mu$ ) denote the corresponding entrance times. (Note that  $e^\mu > 0$  implies  $\phi^\mu(e^\mu(t, x), t, x) = 1$ .) Then the following holds:

$$e^\mu(T, x) \geq T - \frac{1}{c} \quad \text{and} \quad e^\lambda(T, x) \geq T - \frac{1}{c}, \quad \forall x \in [0, 1].$$

Combining this with (3.40) and (2.27), we obtain (3.39).  $\square$

Finally, it is sufficient to extend  $u$  and  $v$  by 0 for  $t \geq T$  to get a global in time solution. The stability property (3.18) follows at once from (3.21)-(3.22), as the r.h.s. in (3.21) and (3.22) tend to 0 as  $(C_1, C_2) \rightarrow (0, 0)$ . The proof of Theorem 1 is complete.  $\square$

#### 4. FINITE TIME STABILIZATION WITH A CONTROL FROM ONE SIDE

In this section, we consider a system of the form

$$\partial_t u + \lambda(u, v) \partial_x u = 0, \quad (t, x) \in (0, +\infty) \times (0, 1), \quad (4.1)$$

$$\partial_t v + \mu(u, v) \partial_x v = 0, \quad (t, x) \in (0, +\infty) \times (0, 1), \quad (4.2)$$

$$u(t, 0) = h(v(t, 0), t), \quad u(0, x) = u_0(x), \quad (4.3)$$

$$v(t, 1) = v_r(t), \quad v(0, x) = v_0(x), \quad (4.4)$$

where  $v_r$  still solves the ODE

$$\frac{d}{dt}v_r(t) = -K \operatorname{sgn}(v_r(t))|v_r(t)|^\gamma, \quad v_r(0) = v_0(1). \quad (4.5)$$

In (4.3),  $h$  denotes some function in  $\mathcal{C}^1([-\overline{C}_1, \overline{C}_1] \times \mathbb{R}^+) \cap W^{1,\infty}((-\overline{C}_1, \overline{C}_1) \times (0, +\infty))$  for some number  $\overline{C}_1 > 0$  such that, for some time  $T_h > 0$ ,

$$h(0, t) = 0 \quad \forall t \geq T_h. \quad (4.6)$$

We introduce the numbers

$$\begin{aligned} C_1 &\in (0, \overline{C}_1], \\ T &:= \frac{1}{c} + \max(T_h, \frac{1}{c} + \frac{C_1^{1-\gamma}}{(1-\gamma)K}), \\ C'_1 &:= \max(C_1, \|h\|_{L^\infty((-C_1, C_1) \times (0, +\infty))}), \\ D_1 &:= \|\partial_v h\|_{L^\infty((-C_1, C_1) \times (0, +\infty))}, \\ D_2 &:= \|\partial_t h\|_{L^\infty((-C_1, C_1) \times (0, +\infty))}, \\ M_1 &:= \max\left(\|\lambda\|_{\mathcal{C}^0([-C'_1, C'_1] \times [-C_1, C_1])}, \|\mu\|_{\mathcal{C}^0([-C'_1, C'_1] \times [-C_1, C_1])}\right), \\ M_2 &:= \max\left(\|\partial_u \mu\|_{\mathcal{C}^0([-C'_1, C'_1] \times [-C_1, C_1])}, \|\partial_v \mu\|_{\mathcal{C}^0([-C'_1, C'_1] \times [-C_1, C_1])}, \right. \\ &\quad \left. \|\partial_u \lambda\|_{\mathcal{C}^0([-C'_1, C'_1] \times [-C_1, C_1])}, \|\partial_v \lambda\|_{\mathcal{C}^0([-C'_1, C'_1] \times [-C_1, C_1])}\right), \\ C_3 &:= \max(\frac{1}{2TM_2}, C_2), \\ C'_3 &:= \max(\frac{KC_1^\gamma}{c}, C_2) \max(1, M_1) \exp(2TM_2C_3). \end{aligned}$$

Note that, if  $\|v\|_{\mathcal{C}^0([0, T] \times [0, 1])} \leq C_1$ , then for all  $t \in (0, T)$

$$|u(t, 0)| \leq C'_1 \quad \text{and} \quad |\partial_t u(t, 0)| \leq D_1 |\partial_t v(t, 0)| + D_2.$$

We shall consider the following conditions

$$C'_3 \leq C_3, \quad (4.7)$$

$$C''_3 := \max(\frac{1}{c}(D_1 C'_3 + D_2), C_2) \max(1, M_1) \exp(2TM_2C_3) \leq C_3. \quad (4.8)$$

Note that (4.7) and (4.8) are satisfied if  $C_1$ ,  $C_2$ , and  $D_2$  are small enough.

We introduce the set

$$\mathcal{D} := \{(u, v) \in \operatorname{Lip}([0, T] \times [0, 1])^2; \|u\|_{\mathcal{C}^0([0, T] \times [0, 1])} \leq C'_1, \|v\|_{\mathcal{C}^0([0, T] \times [0, 1])} \leq C_1, \\ u \text{ is } C_3\text{-Lipschitz, } v \text{ is } C'_3\text{-Lipschitz}\}.$$

We pick a pair  $(u_0, v_0) \in \operatorname{Lip}([0, 1])^2$  fulfilling (3.8)-(3.9) and the following compatibility condition

$$u_0(0) = h(v_0(0), 0). \quad (4.9)$$

Let us do some comments about the boundary condition (4.3). For a system of conservation laws on the interval  $(0, 1)$ , a very general boundary condition at  $x = 0$  takes the form  $f(u(t, 0), v(t, 0)) = 0$ . If  $\partial_u f(u_0, v_0) \neq 0$ , then around  $(u_0, v_0)$  an application of the Implicit Function Theorem gives a relation of the form

$$u(t, 0) = h(v(t, 0))$$

with  $h$  a smooth function of  $v$  in a neighborhood of  $v_0$ . Assume now that the interval represents an edge in a network, and that the left endpoint is a multiple node (i.e. it belongs to at least two edges). The contributions of the other edges at this multiple node can be taken into account in  $h$  through its dependence in  $t$  in (4.3).

We are in a position to state the main result of this section.

**Theorem 2.** *Assume that  $C_1, C_2$  and  $D_2$  are such that the conditions (4.7) and (4.8) are satisfied. Then for any pair  $(u_0, v_0) \in \text{Lip}([0, 1])^2$  fulfilling (3.8), (3.9) and (4.9), there exists a unique solution  $(u, v)$  of (4.1)-(4.5) in the class  $\mathcal{D}$ . Furthermore, the solution is global in time with  $u(t, \cdot) = v(t, \cdot) = 0$  for  $t \geq T$ . Finally, if  $h = h(v)$ , then the equilibrium state  $(0, 0)$  is stable in  $\text{Lip}([0, 1])^2$  for (4.1)-(4.5); that is*

$$\|(u, v)\|_{L^\infty(\mathbb{R}^+; \text{Lip}([0, 1])^2)} \rightarrow 0 \quad \text{as} \quad \|(u_0, v_0)\|_{\text{Lip}([0, 1])^2} \rightarrow 0. \quad (4.10)$$

*Proof.* It is very similar to those of Theorem 1. If  $(\tilde{u}, \tilde{v}) \in \mathcal{D}$  is given, we define  $(u, v) = \mathcal{F}(\tilde{u}, \tilde{v})$  as follows:  $u$  is the weak solution of the system

$$\begin{cases} \partial_t u + \lambda(\tilde{u}, \tilde{v}) \partial_x u = 0, \\ u(t, 0) = h(\tilde{v}(t, 0), t), \quad u(0, x) = u_0(x), \end{cases} \quad \forall (t, x) \in [0, T] \times [0, 1],$$

and  $v$  is the weak solution of the system

$$\begin{cases} \partial_t v + \mu(\tilde{u}, \tilde{v}) \partial_x v = 0, \\ v(t, 1) = v_r(t), \quad v(0, x) = v_0(x), \end{cases} \quad \forall (t, x) \in [0, T] \times [0, 1].$$

Then, using Proposition 2.6 and (4.7)-(4.8), one readily sees that

$$\|u\|_{\mathcal{C}^0([0, T] \times [0, 1])} \leq C'_1, \quad \|v\|_{\mathcal{C}^0([0, T] \times [0, 1])} \leq C_1,$$

$$u \text{ is } C_3''\text{-Lipschitz, hence } u \text{ is } C_3\text{-Lipschitz,} \quad (4.11)$$

$$v \text{ is } C_3'\text{-Lipschitz,} \quad (4.12)$$

so that  $\mathcal{F}$  maps  $\mathcal{D}$  into itself. Let us prove that  $\mathcal{F}$  is continuous,  $\mathcal{D}$  being equipped with the topology of the uniform convergence. Consider a sequence  $\{(\tilde{u}_n, \tilde{v}_n)\} \subset \mathcal{D}$  and a pair  $(\tilde{u}, \tilde{v}) \in \mathcal{D}$  such that

$$\max \left( \|\tilde{u}_n - \tilde{u}\|_{\mathcal{C}^0([0, T] \times [0, 1])}, \|\tilde{v}_n - \tilde{v}\|_{\mathcal{C}^0([0, T] \times [0, 1])} \right) \xrightarrow{n \rightarrow +\infty} 0. \quad (4.13)$$

Let

$$(u_n, v_n) = \mathcal{F}(\tilde{u}_n, \tilde{v}_n) \quad \text{for } n \geq 0, \quad \text{and} \quad (u, v) = \mathcal{F}(\tilde{u}, \tilde{v}). \quad (4.14)$$

We aim to prove that  $u_n \rightarrow u$  and  $v_n \rightarrow v$  uniformly on  $[0, T] \times [0, 1]$  as  $n \rightarrow \infty$ . We focus on  $u_n$ , the argument for  $v_n$  being the same as those given in Lemma 2. We consider the same  $\phi_n, \phi, e_n, e, I_n, I, J_n, J, P_n$  and  $P$ , as in the proof of Lemma 2. Then

$$u_n(t, x) = \begin{cases} h(\tilde{v}_n(e_n(t, x), 0), e_n(t, x)) & \text{if } (t, x) \in J_n, \\ u_0(\phi_n(0, t, x)) & \text{if } (t, x) \in I_n \cup P_n \end{cases}$$

and

$$u(t, x) = \begin{cases} h(\tilde{v}(e(t, x), 0), e(t, x)) & \text{if } (t, x) \in J, \\ u_0(\phi(0, t, x)) & \text{if } (t, x) \in I \cup P. \end{cases}$$

Assume first that  $(t, x) \in J$ . Then  $e(t, x) > 0$  and  $e_n(t, x) > 0$  for  $n$  large enough, by Proposition 2.2. Since  $\tilde{v}_n \rightarrow \tilde{v}$  uniformly on  $[0, T] \times [0, 1]$  and  $e_n(t, x) \rightarrow e(t, x)$ , we infer that

$$u_n(t, x) = h(\tilde{v}_n(e_n(t, x), 0), e_n(t, x)) \rightarrow h(\tilde{v}(e(t, x), 0), e(t, x)) = u(t, x).$$

If now  $(t, x) \in I$ , one can repeat the argument in Lemma 2 to conclude that

$$u_n(t, x) = u_0(\phi_n(0, t, x)) \rightarrow u_0(\phi(0, t, x)) = u(t, x).$$

Thus,  $u_n(t, x) \rightarrow u(t, x)$  for  $(t, x) \in I \cup J$ , hence for a.e.  $(t, x) \in [0, T] \times [0, 1]$ . We have also that  $v_n(t, x) \rightarrow v(t, x)$  for a.e.  $(t, x) \in [0, T] \times [0, 1]$ . We infer from the compactness of  $\mathcal{D}$  in  $\mathcal{C}^0([0, T] \times [0, 1])^2$  that  $(u_n, v_n) \rightarrow (u, v)$  in  $\mathcal{C}^0([0, T] \times [0, 1])^2$ . We conclude with Schauder fixed-point theorem that  $\mathcal{F}$  has a fixed-point  $(u, v) \in \mathcal{D}$ , which is a solution of (4.1)-(4.5) on  $[0, T] \times [0, 1]$ .

Let us now establish the uniqueness of the solution of (4.1)-(4.5) in the class  $\mathcal{D}$ . Assume given two pairs  $(u^1, v^1), (u^2, v^2) \in \mathcal{D}$  of solutions of (4.1)-(4.5); that is, with  $v_r$  defined as in (4.5),  $v^i$ ,  $i = 1, 2$ , is a (weak) solution of

$$\begin{cases} \partial_t v^i + \mu(u^i, v^i) \partial_x v^i = 0, \\ v^i(t, 1) = v_r(t), \quad v^i(0, x) = v_0(x), \end{cases} \quad (t, x) \in (0, T) \times (0, 1),$$

while  $u^i$ ,  $i = 1, 2$ , is a (weak) solution of

$$\begin{cases} \partial_t u^i + \lambda(u^i, v^i) \partial_x u^i = 0, \\ u^i(t, 0) = h(v^i(t, 0), t), \quad u^i(0, x) = u_0(x), \end{cases} \quad (t, x) \in (0, T) \times (0, 1).$$

Let  $\hat{u} = u^1 - u^2$  and  $\hat{v} = v^1 - v^2$ . Note that  $\hat{u}, \hat{v} \in W^{1,\infty}((0, T) \times (0, 1))$  and that  $\hat{u}, \hat{v}$  satisfy

$$\partial_t \hat{u} + \lambda^1 \partial_x \hat{u} + \hat{\lambda} \partial_x u^2 = 0, \tag{4.15}$$

$$\partial_t \hat{v} + \mu^1 \partial_x \hat{v} + \hat{\mu} \partial_x v^2 = 0, \tag{4.16}$$

$$\hat{u}(t, 0) = h(v^1(t, 0), t) - h(v^2(t, 0), t), \tag{4.17}$$

$$\hat{v}(t, 1) = 0, \tag{4.18}$$

$$\hat{u}(0, x) = \hat{v}(0, x) = 0 \tag{4.19}$$

where  $\lambda^i = \lambda(u^i, v^i)$ ,  $\mu^i = \mu(u^i, v^i)$ , and  $\hat{\lambda} = \lambda^1 - \lambda^2$ ,  $\hat{\mu} = \mu^1 - \mu^2$ .

Multiplying in (4.15) by  $2\hat{u}$  and integrating over  $(0, t) \times (0, 1)$  gives

$$\begin{aligned}
\|\hat{u}(t)\|^2 &= -2 \int_0^t \int_0^1 [\lambda^1 \hat{u} \partial_x \hat{u} + \hat{\lambda} \hat{u} \partial_x u^2] dx ds \\
&= \int_0^t \int_0^1 [(\partial_x \lambda^1) |\hat{u}|^2 - 2\hat{\lambda} \hat{u} \partial_x u^2] dx ds - \int_0^t \lambda^1 |\hat{u}|^2|_0^1 ds \\
&\leq 2M_2 C_3 \int_0^t \int_0^1 [|\hat{u}|^2 + |\hat{u}|(|\hat{u}| + |\hat{v}|)] dx ds \\
&\quad + \|\lambda\|_{C^0([-C'_1, C'_1] \times [-C_1, C_1])} D_1^2 \int_0^t |\hat{v}(s, 0)|^2 ds
\end{aligned} \tag{4.20}$$

where we used (3.3).

Multiplying in (4.16) by  $2\hat{v}$  and integrating over  $(0, t) \times (0, 1)$  gives

$$\begin{aligned}
\|\hat{v}(t)\|^2 &= -2 \int_0^t \int_0^1 [\mu^1 \hat{v} \partial_x \hat{v} + \hat{\mu} \hat{v} \partial_x v^2] dx ds \\
&= \int_0^t \int_0^1 [(\partial_x \mu^1) |\hat{v}|^2 - 2\hat{\mu} \hat{v} \partial_x v^2] dx ds + \int_0^t \mu^1 |\hat{v}(s, 0)|^2 ds \\
&\leq 2M_2 C_3 \int_0^t \int_0^1 [|\hat{v}|^2 + |\hat{v}|(|\hat{u}| + |\hat{v}|)] dx ds - c \int_0^t |\hat{v}(s, 0)|^2 ds
\end{aligned} \tag{4.21}$$

where we used (3.3) again. Let us introduce the energy

$$E(t) = \|\hat{u}(t)\|^2 + \|\lambda\|_{C^0([-C'_1, C'_1] \times [-C_1, C_1])} \frac{D_1^2}{c} \|\hat{v}(t)\|^2.$$

Combining (4.20) with (4.21) yields

$$E(t) \leq C \int_0^t E(s) ds,$$

for some  $C$  depending only on  $\mathcal{D}$ , so that  $E \equiv 0$ , by Gronwall's lemma. This proves the uniqueness. For the extinction time, we notice that from the proof of Theorem 1

$$v(t, x) = 0, \quad \text{for } \frac{1}{c} + \frac{C_1^{1-\gamma}}{(1-\gamma)K} \leq t \leq T, \quad 0 \leq x \leq 1.$$

Combined with (4.6), this yields

$$u(t, 0) = h(v(t, 0), t) = 0, \quad \text{for } \max\left(T_h, \frac{1}{c} + \frac{C_1^{1-\gamma}}{(1-\gamma)K}\right) \leq t \leq T.$$

Using (3.3), we conclude that

$$u(T, x) = 0 \quad \forall x \in [0, 1].$$

Assume now that  $h = h(v)$ , i.e.  $D_2 = 0$ . The stability property (4.10) follows at once from (4.6) and (4.7)-(4.8), as  $C'_1 \leq \max(1, D_1)C_1$  and  $(C'_3, C''_3) \rightarrow (0, 0)$  as  $(C_1, C_2) \rightarrow (0, 0)$ . The proof of Theorem 2 is complete.  $\square$

## 5. APPLICATION TO THE REGULATION OF WATER FLOW IN CHANNELS

In this section, we investigate the regulation of water flow in a network of open horizontal channels. We assume that the channels have a rectangular cross section and that the friction on the walls can be neglected. In this context, the flow of the fluid can be described in a satisfactory way by the shallow water equations (also called Saint-Venant equations) (see [20]). The control in feedback form is applied at the vertices of the network, which is assumed to be a tree.

We introduce some notations needed in what follows (we follow closely [9]). Let  $\mathcal{T}$  be a tree, whose vertices (or nodes) are numbered by the index  $n \in \mathcal{N} = \{1, \dots, N\}$ , and whose edges are numbered by the index  $i \in \mathcal{I} = \{1, \dots, I\}$  with  $I = N - 1$ . We choose a simple vertex, called the *root* of  $\mathcal{T}$  and denoted by  $\mathcal{R}$ , and which corresponds to the index  $n = N$ . We choose an orientation of the edges in the tree such that  $\mathcal{R}$  is the “last” encountered vertex. It is similar to those of a fluvial network in which each edge stands for a river, and  $\mathcal{R}$  indicates the place where the last river enters into the ocean.

We denote by  $l_i$  the length of the edge with index  $i$ . Once the orientation is chosen, each point of the  $i$ -th edge is identified with a real number  $x \in [0, l_i]$ . The points  $x = 0$  and  $x = l_i$  are termed the *initial point* and the *final point* of the  $i$ -edge, respectively.

Renumbering the edges if needed, we may assume that the edge with index  $i$  has as initial point the vertex with the (same) index  $n = i$  for all  $i \in \mathcal{I}$ .

We denote by  $\mathcal{I}_n \subset \mathcal{I}$ ,  $n = 1, \dots, N$ , the set of indices of those edges having the vertex of index  $n$  as one of their ends. Let

$$\varepsilon_{i,n} = \begin{cases} 0 & \text{if the vertex with index } n \text{ is the initial point of the edge with index } i; \\ 1 & \text{if the vertex with index } n \text{ is the final point of the edge with index } i. \end{cases}$$

Note that  $\varepsilon_{i,i} = 0$  for all  $i \in \mathcal{I}$ , and that  $\varepsilon_{N-1,N} = 1$ . A node with index  $n$  is said to be *simple* (resp. *multiple*) if  $\#(\mathcal{I}_n) = 1$  (resp.  $\#(\mathcal{I}_n) \geq 2$ ). The sets of indices of simple and multiple nodes are denoted by  $\mathcal{N}_S$  and  $\mathcal{N}_M$ , respectively. The *depth* of the tree is the greater number of edges in a path from one simple node to  $\mathcal{R}$ .

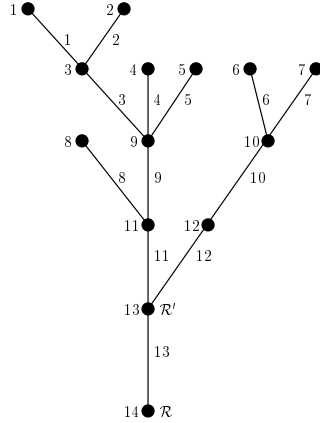


FIGURE 2. A tree with 14 nodes, a depth equal to 5, with simple nodes  $\mathcal{N}_S = \{1, 2, 4, 5, 6, 7, 8, 14\}$  and multiple nodes  $\mathcal{N}_M = \{3, 9, 10, 11, 12, 13\}$ .

Pick any channel represented by (say) the  $i$ -th edge of the tree, which is identified with the segment  $[0, l_i]$ . Then the shallow water equations read

$$\partial_t H_i + \partial_x (H_i V_i) = 0, \quad t > 0, \quad 0 < x < l_i, \quad (5.1)$$

$$\partial_t V_i + \partial_x \left( \frac{V_i^2}{2} + g H_i \right) = 0, \quad t > 0, \quad 0 < x < l_i, \quad (5.2)$$

where  $H_i(t, x)$  (resp.  $V_i(t, x)$ ) is the water depth (resp. the water velocity) along the  $i$ -th channel, and  $g$  is the gravitation constant. The equations (5.1)-(5.2) have to be supplemented with some initial conditions

$$H_i(0, x) = H_{i,0}(x), \quad V_i(0, x) = V_{i,0}(x), \quad 0 < x < l_i \quad (5.3)$$

and with two boundary conditions. In general, there are at the two ends of the channel (i.e. at  $x = 0$  and at  $x = l_i$ ) some hydraulic devices to assign the values of the flow rate. Recall that the flow rate is defined along the channel as

$$Q_i(t, x) := H_i(t, x) V_i(t, x).$$

At any multiple node  $n \in \mathcal{N}_M$ , the equation of conservation of the flow

$$\sum_{i \in \mathcal{J}_n} (-1)^{\varepsilon_{i,n}} Q_i(t, \varepsilon_{i,n} l_i) = 0 \quad (5.4)$$

has to be taken into consideration. It yields a boundary condition (coming from the physics) in which no control applies. Let  $i_0 \in \mathcal{J}_n$  be the only index such that  $\varepsilon_{i_0,n} = 0$ , namely  $i_0 = n$ . Then (5.4) can be written

$$Q_{i_0}(t, 0) = \sum_{i \in \mathcal{J}_n, i \neq i_0} Q_i(t, l_i). \quad (5.5)$$

Thus, the flow rate may be controlled at the final points of the edges of indices  $i \neq i_0$ , while it is prescribed by (5.5) at the initial point of the edge of index  $i_0$ .

We aim to stabilize the system around some equilibrium state, represented by a sequence  $\{(H_i^*, V_i^*)\}_{1 \leq i \leq I}$  of pairs of positive numbers. Let  $Q_i^* = H_i^* V_i^*$ . For (5.4) to be valid as  $t \rightarrow \infty$ , we impose that

$$\sum_{i \in \mathcal{J}_n} (-1)^{\varepsilon_{i,n}} Q_i^* = 0, \quad \forall n \in \mathcal{N}_M. \quad (5.6)$$

Introduce the characteristic velocities

$$\mu_i = V_i - \sqrt{g H_i}, \quad (5.7)$$

$$\lambda_i = V_i + \sqrt{g H_i} \quad (5.8)$$

and the Riemann invariants (see [13, 20])

$$u_i = V_i + 2\sqrt{g H_i} - (V_i^* + 2\sqrt{g H_i^*}), \quad (5.9)$$

$$v_i = V_i - 2\sqrt{g H_i} - (V_i^* - 2\sqrt{g H_i^*}). \quad (5.10)$$

We shall assume thereafter that the flow is *subcritical* or *fluvial*; that is, the characteristic velocities are of opposite sign

$$\mu_i < 0 < \lambda_i.$$



Clearly, this holds if

$$0 < V_i^* < \sqrt{gH_i^*} \quad (5.11)$$

and  $\max(|H_i - H_i^*|, |V_i - V_i^*|)$  is small enough. From now on, we assume that (5.11) holds for all  $i \in \mathcal{I}$ , and we pick a number  $c > 0$  such that

$$\sqrt{gH_i^*} - V_i^* > 2c, \quad \forall i \in \mathcal{I}. \quad (5.12)$$

Note that (5.9)-(5.10) may be inverted as

$$H_i = \left( \sqrt{H_i^*} + \frac{1}{4\sqrt{g}}(u_i - v_i) \right)^2, \quad (5.13)$$

$$V_i = V_i^* + \frac{1}{2}(u_i + v_i). \quad (5.14)$$

Substituting the values of  $H_i, V_i$  in (5.9)-(5.10) yields

$$\mu_i = V_i^* - \sqrt{gH_i^*} + \frac{1}{4}(u_i + 3v_i), \quad (5.15)$$

$$\lambda_i = V_i^* + \sqrt{gH_i^*} + \frac{1}{4}(3u_i + v_i). \quad (5.16)$$

Combined with (5.12), this shows that

$$\max(|u_i|, |v_i|) \leq c \quad \Rightarrow \quad \mu_i < -c < c < \lambda_i.$$

The shallow water equations (5.1)-(5.2), when expressed in terms of the Riemann invariants  $u_i$  and  $v_i$ , read

$$\partial_t u_i + \lambda_i(u_i, v_i) \partial_x u_i = 0, \quad t > 0, \quad 0 < x < l_i, \quad (5.17)$$

$$\partial_t v_i + \mu_i(u_i, v_i) \partial_x v_i = 0, \quad t > 0, \quad 0 < x < l_i. \quad (5.18)$$

Let us now turn our attention to the boundary conditions. Consider first a boundary condition associated with an active control, e.g.

$$\frac{d}{dt} v_i(t, l_i) = -K \operatorname{sgn}(v_i(t, l_i)) |v_i(t, l_i)|^\gamma. \quad (5.19)$$

In practice, one would like to assign the value of  $Q_i(t, l_i) = H_i(t, l_i)V_i(t, l_i)$  by using the output  $H_i(t, l_i)$  only. Using (5.10), it is sufficient to set

$$Q_i(t, l_i) = H_i(t, l_i) \left( v_i(t, l_i) + 2\sqrt{gH_i(t, l_i)} + V_i^* - 2\sqrt{gH_i^*} \right), \quad (5.20)$$

where  $v_i$  solves (5.19) together with the initial condition

$$v_i(0, l_i) = V_i(0, l_i) - 2\sqrt{gH_i(0, l_i)} - V_i^* + 2\sqrt{gH_i^*}. \quad (5.21)$$

For a control applied to the initial point of the  $i$ -edge, we set

$$Q_i(t, 0) = H_i(t, 0) \left( u_i(t, 0) - 2\sqrt{gH_i(t, 0)} + V_i^* + 2\sqrt{gH_i^*} \right), \quad (5.22)$$

where  $u_i(\cdot, 0)$  solves

$$\frac{d}{dt} u_i(t, 0) = -K \operatorname{sgn}(u_i(t, 0)) |u_i(t, 0)|^\gamma, \quad (5.23)$$

$$u_i(0, 0) = V_i(0, 0) + 2\sqrt{gH_i(0, 0)} - V_i^* - 2\sqrt{gH_i^*}. \quad (5.24)$$

Consider next a boundary condition without any active control. For a simple node  $n \in \mathcal{N}_S$  and the corresponding edge  $i \in \mathcal{J}_n$ , if  $\varepsilon_{i,n} = 0$  (i.e. the node  $n$  is the initial point of the edge  $i$ ), then  $n = i$  and a natural boundary condition at the node  $n$  is given by the relation

$$Q_i(t, 0) = Q_i^*, \quad (5.25)$$

that is

$$F_i(u_i(t, 0), v_i(t, 0)) = 0 \quad (5.26)$$

where

$$F_i(u, v) = (\sqrt{H_i^*} + \frac{1}{4\sqrt{g}}(u - v))^2 (V_i^* + \frac{1}{2}(u + v)) - H_i^* V_i^*.$$

Since

$$F_i(0, 0) = 0 \text{ and } \frac{\partial F_i}{\partial u}(0, 0) = \frac{1}{2}\sqrt{H_i^*}(\sqrt{H_i^*} + \frac{V_i^*}{\sqrt{g}}) > 0$$

it follows from the Implicit Function Theorem that there exist a number  $\delta_i > 0$  and a function  $h_i \in C^1(\mathbb{R})$  with  $h_i(0) = 0$  such that for  $\max(|u|, |v|) < \delta_i$ ,

$$F_i(u, v) = 0 \iff u = h_i(v).$$

Thus (5.26) may be written, at least locally, in the form

$$u_i(t, 0) = h_i(v_i(t, 0)).$$

Finally, for a multiple node  $n \in \mathcal{N}_M$ , if  $i_0 \in \mathcal{J}_n$  is the only index such that  $\varepsilon_{i_0, n} = 0$  (i.e.  $i_0 = n$ ), then (5.5) may be written

$$F_{i_0}(U_{i_0}(t, 0), v_{i_0}(t, 0), U(t), V(t)) = 0$$

where  $U(t) = (u_i(t, l_i))_{i \in \mathcal{J}_n, i \neq i_0}$ ,  $V(t) = (v_i(t, l_i))_{i \in \mathcal{J}_n, i \neq i_0}$  and

$$\begin{aligned} F_{i_0}(u_{i_0}, v_{i_0}, U, V) &= (\sqrt{H_{i_0}^*} + \frac{1}{4\sqrt{g}}(u_{i_0} - v_{i_0}))^2 (V_{i_0}^* + \frac{1}{2}(u_{i_0} + v_{i_0})) \\ &\quad - \sum_{i \in \mathcal{J}_n, i \neq i_0} (\sqrt{H_i^*} + \frac{1}{4\sqrt{g}}(u_i - v_i))^2 (V_i^* + \frac{1}{2}(u_i + v_i)). \end{aligned}$$

Note that, by (5.6),  $F_{i_0}(0, 0, 0, 0) = 0$  and

$$\frac{\partial F_{i_0}}{\partial u_{i_0}}(0, 0, 0, 0) = \frac{1}{2}\sqrt{H_{i_0}^*}(\sqrt{H_{i_0}^*} + \frac{V_{i_0}^*}{\sqrt{g}}) > 0.$$

We may pick a number  $\delta_{i_0} > 0$  and a function  $H_{i_0}$  of class  $\mathcal{C}^1$  around 0 such that, if  $|u_i| < \delta_{i_0}$  and  $|v_i| < \delta_{i_0}$  for all  $i \in \mathcal{J}_n$ , we have

$$F_{i_0}(u_{i_0}, v_{i_0}, U, V) = 0 \iff u_{i_0} = H_{i_0}(v_{i_0}, U, V).$$

Replacing  $(u_i, v_i)$  by  $(u_i(t, l_i), v_i(t, l_i))$  for  $i \neq i_0$  in  $U, V$ , we see that (5.5) may be written, at least locally, in the form

$$u_{i_0}(t, 0) = h_{i_0}(v_{i_0}(t, 0), t) \quad (5.27)$$

where  $h_{i_0} \in \mathcal{C}^1(\mathbb{R}^2)$  and  $h_{i_0}(0, t) = 0$  if  $u_i(t, l_i) = v_i(t, l_i) = 0$  for all  $i \in \mathcal{J}_n \setminus \{i_0\}$ .

We are in a position to state our results for the regulation of water flow in channels. Consider first one channel ( $\mathcal{N} = \{1, 2\}$ ,  $\mathcal{J} = \{1\}$ ) represented by the segment  $[0, l_1]$ .

**Theorem 3.** *(Two boundary controls) Assume that (5.11) holds for  $i = 1$ , and pick any  $c > 0$  as in (5.12). Then there exists a number  $\delta > 0$  such that for all  $(H_{1,0}, V_{1,0}) \in \text{Lip}([0, l_1])^2$  with*

$$\max(\|H_{1,0} - H_1^*\|_{W^{1,\infty}(0,l_1)}, \|V_{1,0} - V_1^*\|_{W^{1,\infty}(0,l_1)}) < \delta, \quad (5.28)$$

*there exists for any  $T > 0$  a unique solution  $(H_1, V_1) \in \text{Lip}([0, T] \times [0, l_1])^2$  of (5.1)-(5.3) and (5.19)-(5.24). Furthermore, there exists a function  $t^*(H_1^*, V_1^*, c, \delta, K, \gamma)$  with  $\lim_{\delta \rightarrow 0} t^* = 0$  such that*

$$H_1(t, x) = H_1^*, \quad V_1(t, x) = V_1^* \quad t \geq \frac{l_1}{c} + t^*, \quad x \in (0, l_1). \quad (5.29)$$

*Finally, the equilibrium point  $(H_1^*, V_1^*)$  is stable in  $\text{Lip}([0, 1])^2$  for the system (5.1)-(5.3) and (5.19)-(5.24).*

*Proof.* Noticing that the map  $\Theta : (H_1, V_1) \rightarrow (u_1, v_1)$  defined along (5.9)-(5.10) is locally around  $(H_1^*, V_1^*)$  a diffeomorphism of class  $\mathcal{C}^\infty$ , the condition (5.28) implies (3.8)-(3.9) for  $C_1$  and  $C_2$  as in Theorem 1 (applied actually on the interval  $(0, l_1)$  rather than  $(0, 1)$ ), provided that  $\delta < \delta_0$  is small enough. We modify the functions  $\mu_1(u, v)$  and  $\lambda_1(u, v)$  outside  $[-c, c]^2$  so that

$$\mu_1(u, v) \leq -c < c \leq \lambda_1(u, v), \quad (u, v) \in \mathbb{R}^2.$$

Let  $(u_1, v_1)$  be the solution given by Theorem 1, and let  $(H_1, V_1) := \Theta^{-1}(u_1, v_1)$ . If  $C_1$  is chosen sufficiently small, then we infer from (3.21) that

$$\begin{aligned} \max(|u_1(t, x)|, |v_1(t, x)|) &\leq C_1 < c, & t \geq 0, \quad 0 < x < l_1, \\ \max(|H_1(t, x) - H_1^*|, |V_1(t, x) - V_1^*|) &< \delta, & t \geq 0, \quad 0 < x < l_1. \end{aligned}$$

It follows that for all  $T > 0$ ,  $(H_1, V_1) \in \text{Lip}([0, T] \times [0, l_1])^2$  is a solution of (5.1)-(5.3) and (5.19)-(5.24) such that (5.29) holds with  $t^* = C_1^{1-\gamma}/((1-\gamma)K)$ . Note that the range of  $C_1$  in Theorem 1 depends on  $H_1^*$ ,  $V_1^*$  and  $c$  through the constants  $M_1$  and  $M_2$ , and that  $(C_1, C_2) \rightarrow (0, 0)$  as  $\delta \rightarrow 0$ . Thus  $t^* \rightarrow 0$  as  $\delta \rightarrow 0$  with  $K$  and  $\gamma$  kept constant. The uniqueness of  $(H_1, V_1)$  in the class  $\text{Lip}([0, T] \times [0, l_1])^2$  for all  $T > 0$  follows at once from those of  $(u_1, v_1)$  in the same class, as stated in Theorem 1. The stability property follows from (3.18).  $\square$

If the control is active at one endpoint of the channel only, a finite-time stabilization may be derived as well.

**Theorem 4.** *(One boundary control) Assume that (5.11) holds for  $i = 1$ , and pick any  $c > 0$  as in (5.12). Then there exists a number  $\delta > 0$  such that for all  $(H_{1,0}, V_{1,0}) \in \text{Lip}([0, l_1])^2$  with*

$$\max(\|H_{1,0} - H_1^*\|_{W^{1,\infty}(0,l_1)}, \|V_{1,0} - V_1^*\|_{W^{1,\infty}(0,l_1)}) < \delta, \quad (5.30)$$

$$H_{1,0}(l_1)V_{1,0}(l_1) = H_1^*V_1^*, \quad (5.31)$$

*there exists for any  $T > 0$  a unique solution  $(H_1, V_1) \in \text{Lip}([0, T] \times [0, l_1])^2$  of (5.1)-(5.3), (5.19)-(5.21) and (5.25). Furthermore, there exists a function  $t^*(H_1^*, V_1^*, c, \delta, K, \gamma)$  with  $\lim_{\delta \rightarrow 0} t^* = 0$  such that*

$$H_1(t, x) = H_1^*, \quad V_1(t, x) = V_1^* \quad t \geq \frac{2l_1}{c} + t^*, \quad x \in (0, l_1). \quad (5.32)$$

*Finally, the equilibrium point  $(H_1^*, V_1^*)$  is stable in  $\text{Lip}([0, 1])^2$  for the system (5.1)-(5.3), (5.19)-(5.21) and (5.25).*

*Proof.* It is sufficient to proceed as for the proof of Theorem 3, and to use Theorem 2 (on the domain  $[0, l_1]$  and with a control active at the final point only).  $\square$

A direct application of Theorem 4 gives the following result for a chain of two channels ( $\mathcal{N} = \{1, 2, 3\}$ ,  $\mathcal{J} = \{1, 2\}$ ), for which there is no active control at the internal node.

**Corollary 1.** *(Two channels and two controls) Assume that (5.11) holds for  $i = 1, 2$  with  $Q_1^* = Q_2^*$ , and pick any  $c > 0$  as in (5.12). Then there exists a number  $\delta > 0$  such that for all  $(H_{1,0}, V_{1,0}, H_{2,0}, V_{2,0}) \in \text{Lip}([0, l_1])^2 \times \text{Lip}([0, l_2])^2$  with*

$$\max(\|H_{i,0} - H_i^*\|_{W^{1,\infty}(0,l_i)}, \|V_{i,0} - V_i^*\|_{W^{1,\infty}(0,l_i)}) < \delta, \quad i = 1, 2, \quad (5.33)$$

$$H_{1,0}(l_1)V_{1,0}(l_1) = Q_1^* = Q_2^* = H_{2,0}(0)V_{2,0}(0), \quad (5.34)$$

*there exists for any  $T > 0$  a unique solution  $(H_i, V_i) \in \text{Lip}([0, T] \times [0, l_1])^2$  of (5.1)-(5.3) for  $i = 1, 2$ , (5.19)-(5.21) for  $i = 2$ , (5.22)-(5.24) for  $i = 1$ , and*

$$Q_1(t, l_1) = Q_1^* = Q_2^* = Q_2(t, 0), \quad t > 0.$$

*Furthermore, there exists a function  $t^*(H_1^*, V_1^*, H_2^*, V_2^*, c, \delta, K, \gamma)$  with  $\lim_{\delta \rightarrow 0} t^* = 0$  such that*

$$H_i(t, x) = H_i^*, \quad V_i(t, x) = V_i^* \quad t \geq \frac{2 \max(l_1, l_2)}{c} + t^*, \quad x \in (0, l_i), \quad i = 1, 2. \quad (5.35)$$

We can extend the above results to a network of open channels which is a tree. We assume that the incoming flows can be controlled at each multiple node (the outgoing flow being uncontrolled and deduced from the conservation of the flows). In terms of Riemann invariants, for the edge with index  $i$ , the function  $v_i$  is controlled at  $x = l_i$  according to (5.19), while the function  $u_i$  is controlled at  $x = 0$  according to (5.23) only if the initial point of the edge is a simple node (otherwise,  $u_i(t, 0)$  is given by (5.27)).

The main result of this section is the following

**Theorem 5.** *(Network of open channels) Consider a tree with  $N$  nodes and  $I = N - 1$  edges. Assume that (5.11) holds for  $i = 1, \dots, I$ , that (5.6) holds, and pick any  $c > 0$  as in (5.12). Then there exists a number  $\delta > 0$  such that for all  $(H_{1,0}, V_{1,0}, \dots, H_{I,0}, V_{I,0}) \in \text{Lip}([0, l_1])^2 \times \dots \times \text{Lip}([0, l_I])^2$  with*

$$\max(\|H_{i,0} - H_i^*\|_{W^{1,\infty}(0,l_i)}, \|V_{i,0} - V_i^*\|_{W^{1,\infty}(0,l_i)}) < \delta, \quad i = 1, \dots, I, \quad (5.36)$$

$$H_{n,0}(0,0)V_{n,0}(0,0) = \sum_{i \in \mathcal{J}_n, i \neq n} H_{i,0}(0, l_i)V_{i,0}(0, l_i), \quad \forall n \in \mathcal{N}_M, \quad (5.37)$$

*there exists for any  $T > 0$  a unique function  $(H_1, V_1, \dots, H_I, V_I) \in \text{Lip}([0, T] \times [0, l_1])^2 \times \dots \times \text{Lip}([0, T] \times [0, l_I])^2$  such that, for all  $i = 1, \dots, I$ , (5.1)-(5.3) and (5.19)-(5.21) hold, and (5.22)-(5.24) hold if the initial point of the  $i$ -th edge is simple, while (5.5) holds if the initial point of the  $i$ -th edge is multiple. Furthermore, there exists a function  $t^*(H_1^*, V_1^*, \dots, H_I^*, V_I^*, \delta, c, K, \gamma)$  with  $\lim_{\delta \rightarrow 0} t^* = 0$  such that*

$$H_i(t, x) = H_i^*, \quad V_i(t, x) = V_i^* \quad t \geq \frac{p \max_{1 \leq i \leq I} l_i}{c} + t^*, \quad x \in (0, l_i), \quad i = 1, \dots, I, \quad (5.38)$$

*where  $p$  denotes the depth of the tree. Finally, the equilibrium state  $(H_i^*, V_i^*)_{1 \leq i \leq I}$  is stable in  $\text{Lip}([0, l_1])^2 \times \dots \times \text{Lip}([0, l_I])^2$  for the system.*

*Proof.* The proof is done by induction on the number of edges  $I \geq 1$ . For  $I = 1$ , the result was already proved in Theorem 3. Note that the norm  $\|(u_1, v_1)\|_{L^\infty(\mathbb{R}^+; \text{Lip}([0, l_1])^2)}$  in Theorem 3 is as small as desired if  $\delta$  is small enough. Let  $I \geq 2$ , and assume the result true for any tree with at most  $I - 1$  edges, with the norms  $\|(u_i, v_i)\|_{L^\infty(\mathbb{R}^+; \text{Lip}([0, l_i])^2)}$  in the edges of the tree as small as desired if  $\delta$  is small enough. Pick any tree with  $I$  edges. Recall that the root  $\mathcal{R}$  is the node with index  $N$ , and that it is the final point of the edge of index  $I = N - 1$ . Denote by  $\mathcal{R}'$  the initial point of the edge of index  $I$ , i.e. the node of index  $N - 1$ . Let  $k = \#(\mathcal{J}_{N-1})$ , and let us denote by  $\mathcal{T}_1, \dots, \mathcal{T}_{k-1}$  the subtrees of  $\mathcal{T}$  with  $\mathcal{R}'$  as root. ( $\mathcal{R}$  does not belong to any of them.) Note that the subsystem associated with any subtree  $\mathcal{T}_i$  is decoupled from the other subtrees and from the last edge of index  $I$ . An application of the induction hypothesis on each subtree  $\mathcal{T}_i$ ,  $1 \leq i \leq k - 1$ , yields the existence (and uniqueness) of the functions  $(H_i, V_i)$  for  $i = 1, \dots, I - 1$ . Next, the existence and uniqueness of  $(H_I, V_I)$  follows at once from Theorem 2. Indeed, the constant  $D_2$  in Theorem 2 may be taken as small as we want if  $\delta$  is sufficiently small, for the quantities  $\|\partial_t u_i(\cdot, l_i)\|_\infty$  and  $\|\partial_t v_i(\cdot, l_i)\|_\infty$  for  $i \in \mathcal{J}_{N-1} \setminus \{N - 1\}$  may be taken arbitrarily small by the induction assumption. Furthermore, the norm  $\|(u_I, v_I)\|_{L^\infty(\mathbb{R}^+; \text{Lip}([0, l_I])^2)}$  tends to 0 with  $\delta$ , by (4.11)-(4.12). The condition (5.38) is obtained by an obvious induction on the depth of the tree.  $\square$

#### APPENDIX: PROOF OF PROPOSITION 2.1.

First, we introduce some extension operator  $\Pi$  which maps a function  $a : [0, T] \times [0, 1] \rightarrow \mathbb{R}$  to a function  $\tilde{a} = \Pi(a) : \mathbb{R}^2 \rightarrow \mathbb{R}$  defined as follows:

- for  $0 \leq t \leq T$

$$\tilde{a}(t, x) = \begin{cases} a(t, x) & \text{if } x \in [0, 1], \\ a(t, 2 - x) & \text{if } x \in [1, 2], \end{cases}$$

and  $\tilde{a}(t, x)$  is 2-periodic in  $x$  (i.e.  $\tilde{a}(t, x + 2) = \tilde{a}(t, x)$ );

- for  $t > T$ ,  $\tilde{a}(t, x) = \tilde{a}(T, x)$  for all  $x \in \mathbb{R}$ ;
- for  $t < 0$ ,  $\tilde{a}(t, x) = \tilde{a}(0, x)$  for all  $x \in \mathbb{R}$ ;

It is easy to see that  $\Pi$  is a (linear) operator from  $\mathcal{C}^0([0, T] \times [0, 1])$  to  $\mathcal{C}^0(\mathbb{R}^2)$  (resp. from  $L^\infty(0, T; \text{Lip}([0, 1]))$  to  $L^\infty(\mathbb{R}; \text{Lip}(\mathbb{R}))$ ) such that

$$\|\Pi(a)\|_{L^\infty(\mathbb{R}^2)} = \|a\|_{\mathcal{C}^0([0, T] \times [0, 1])}, \quad (5.39)$$

$$\|\Pi(a)\|_{L^\infty(\mathbb{R}; \text{Lip}(\mathbb{R}))} = \|a\|_{L^\infty(0, T; \text{Lip}([0, 1]))}, \quad (5.40)$$

$$\Pi(a)(t, x) \geq c \quad \forall (t, x) \in \mathbb{R}^2 \quad \text{if} \quad a(t, x) \geq c \quad \forall (t, x) \in [0, T] \times [0, 1]. \quad (5.41)$$

Let  $a$  fulfill (2.2), and let  $\phi$  (resp.  $\tilde{\phi}$ ) denote the flow associated with  $a$  (resp. with  $\tilde{a} = \Pi(a)$ ). Then  $\tilde{\phi}$  is defined in  $\mathbb{R}^3$ , and

$$\phi(s, t, x) = \tilde{\phi}(s, t, x) \quad \forall (s, t, x) \in \text{Dom } \phi. \quad (5.42)$$

Thus it is sufficient to prove that  $\tilde{\phi}$  is  $K$ -Lipschitz on  $[0, T]^2 \times [0, 1]$ . To this end, pick any  $(s_1, t_1, x_1), (s_2, t_2, x_2) \in [0, T]^2 \times [0, 1]$ . Then

$$\begin{aligned} & |\tilde{\phi}(s_1, t_1, x_1) - \tilde{\phi}(s_2, t_2, x_2)| \\ & \leq |\tilde{\phi}(s_1, t_1, x_1) - \tilde{\phi}(s_2, t_1, x_1)| + |\tilde{\phi}(s_2, t_1, x_1) - \tilde{\phi}(s_2, t_2, x_1)| + |\tilde{\phi}(s_2, t_2, x_1) - \tilde{\phi}(s_2, t_2, x_2)| \\ & =: I_1 + I_2 + I_3. \end{aligned}$$

First,

$$I_1 = \left| \int_{s_1}^{s_2} \partial_s \tilde{\phi}(\tau, t_1, x_1) d\tau \right| = \left| \int_{s_1}^{s_2} \tilde{a}(\tau, \tilde{\phi}(\tau, t_1, x_1)) d\tau \right| \leq \|a\|_{\mathcal{C}^0([0, T] \times [0, 1])} |s_1 - s_2|, \quad (5.43)$$

where we used (5.39). For  $I_2$ , we notice that for all  $s$

$$\begin{aligned} |\partial_s [\tilde{\phi}(s, t_1, x_1) - \tilde{\phi}(s, t_2, x_1)]| &= |\tilde{a}(s, \tilde{\phi}(s, t_1, x_1)) - \tilde{a}(s, \tilde{\phi}(s, t_2, x_1))| \\ &\leq L |\tilde{\phi}(s, t_1, x_1) - \tilde{\phi}(s, t_2, x_1)| \end{aligned}$$

where we used (5.40). Gronwall's lemma combined to the estimate for  $I_1$  yields then

$$\begin{aligned} |\tilde{\phi}(s_2, t_1, x_1) - \tilde{\phi}(s_2, t_2, x_1)| &\leq |\tilde{\phi}(t_2, t_1, x_1) - \tilde{\phi}(t_2, t_2, x_1)| e^{L|s_2 - t_2|} \\ &\leq |\tilde{\phi}(t_2, t_1, x_1) - \tilde{\phi}(t_1, t_1, x_1)| e^{LT} \\ &\leq \|a\|_{\mathcal{C}^0([0, T] \times [0, 1])} e^{LT} |t_1 - t_2|. \end{aligned} \quad (5.44)$$

Finally, for  $I_3$ , we notice that for all  $s$

$$\begin{aligned} |\partial_s [\tilde{\phi}(s, t_2, x_1) - \tilde{\phi}(s, t_2, x_2)]| &= |\tilde{a}(s, \tilde{\phi}(s, t_2, x_1)) - \tilde{a}(s, \tilde{\phi}(s, t_2, x_2))| \\ &\leq L |\tilde{\phi}(s, t_2, x_1) - \tilde{\phi}(s, t_2, x_2)|, \end{aligned}$$

which, combined with Gronwall lemma, yields

$$|\tilde{\phi}(s_2, t_2, x_1) - \tilde{\phi}(s_2, t_2, x_2)| \leq e^{L|s_2 - t_2|} |x_1 - x_2| \leq e^{LT} |x_1 - x_2|. \quad (5.45)$$

Then (2.6) follows at once from (5.43)-(5.45).

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#### REFERENCES

- [1] F. Alabau-Boussouira, V. Perrollaz, L. Rosier, *Finite-time stabilization of a network of strings*, in preparation.
- [2] A. Bacciotti, L. Rosier, *Liapunov functions and stability in control theory*. Second edition. Communications and Control Engineering Series. Springer-Verlag, Berlin, 2005.
- [3] G. Bastin, J.-M. Coron, *On boundary feedback stabilization of non-uniform linear  $2 \times 2$  hyperbolic systems over a bounded interval*, Systems Control Lett. **60** (2011), no. 11, 900906.
- [4] G. Bastin, J.-M. Coron, and B. d'Andréa-Novel, *Dissipative boundary conditions for one dimensional nonlinear hyperbolic systems*, SIAM Journal of Control and Optimization **47** (2008), No. 3, 1460–1498.
- [5] S.P. Bhat, D.S. Bernstein, *Finite-time stability of continuous autonomous systems*, SIAM J. Control Optim. **38** (2000), no. 3, 751–766.
- [6] R. Carles, C. Gallo, *Finite time extinction by nonlinear damping for the Schrödinger equation*, Comm. Partial Differential Equations **36** (2011), no. 6, 961–975.

- [7] J.-M. Coron, B. d'Andréa-Novel, G. Bastin, *A Lyapunov approach to control irrigation canals modeled by Saint-Venant equations*, in CD-Rom proceedings, Paper F1008-5 ECC 99 (1999), Karlsruhe, Germany.
- [8] S. Cox, E. Zuazua, *The rate at which energy decays in a string damped at one end*, Indiana Univ. Math. J. 44 (1995), no. 2, 545–573.
- [9] R. Dáger, E. Zuazua, *Controllability of tree-shaped networks of vibrating strings*, C. R. Acad. Sci. Paris Sér. I Math. **332** (2001), no. 12, 1087–1092.
- [10] R. Dáger, E. Zuazua, *Wave propagation, observation and control in 1-d flexible multi-structures*. Mathématiques & Applications (Berlin) [Mathematics & Applications], 50. Springer-Verlag, Berlin, 2006.
- [11] J. Daz, *Special finite time extinction in nonlinear evolution systems: dynamic boundary conditions and Coulomb friction type problems*. Nonlinear elliptic and parabolic problems, 71–97, Progr. Nonlinear Differential Equations Appl., 64, Birkhuser, Basel, 2005.
- [12] V. Dos Santos, G. Bastin, J.-M. Coron, B. d'Andra-Novel, *Boundary control with integral action for hyperbolic systems of conservation laws: stability and experiments*, Automatica J. IFAC 44 (2008), no. 5, 1310–1318.
- [13] L. C. Evans, *Partial differential equations*. Graduate Studies in Mathematics, 19. American Mathematical Society, Providence, RI, 1998.
- [14] O. Glass, *On the controllability of the 1-D isentropic Euler equation*, J. Eur. Math. Soc. (JEMS) 9 (2007), no. 3, 427–486.
- [15] J. M. Greenberg, T.-t. Li, *The effect of boundary damping for the quasilinear wave equations*, Journal of Differential Equations **52** (1984), 66–75.
- [16] M. Gugat, M. Herty, *Existence of classical solutions and feedback stabilisation for the flow in gas networks*, ESAIM: Control, Optimisation and Calculus of Variations **17** (2011), No. 1, 28–51.
- [17] M. Gugat, G. Leugering, *Global boundary controllability of the de St. Venant equations between steady states*, Ann. Inst. H. Poincaré Anal. Non Linéaire **20** (2003), no. 1, 1–11.
- [18] M. Gugat, G. Leugering, *Global boundary controllability of the Saint-Venant system for sloped canals with friction*, Ann. Inst. H. Poincaré Anal. Non Linéaire **26** (2009), no. 1, 257–270.
- [19] M. Gugat, M. Dick, G. Leugering, *Gas flow in fan-shaped networks: classical solutions and feedback stabilization*, SIAM J. Control Optim. **49** (2011), no. 5, 2101–2117.
- [20] J. de Halleux, C. Prieur, J.-M. Coron, B. d'Andréa-Novel, and G. Bastin, *Boundary feedback control in network of open channels*, Automatica **39** (2003), 1365–1376.
- [21] P. Hartman, *Ordinary Differential Equations*, John Wiley & Sons, Inc., New York-London-Sidney, 1964.
- [22] V. Komornik, *Rapid boundary stabilization of the wave equation*, SIAM J. Control Opt. **29** (1991), 197–208.
- [23] V. Komornik, *Exact Controllability and Stabilization. The Multiplier Method*. Research in Applied Mathematics, Masson, Wiley, 1994.
- [24] C. Laurent, L. Rosier, B.-Z. Zhang, *Control and stabilization of the Korteweg-de Vries equation on a periodic domain*, Comm. Partial Differential Equations 35 (2010), no. 4, 707–744.
- [25] G. Leugering, J.-P. G. Schmidt, *On the modelling and stabilization of flows in networks of open canals*, SIAM Journal of Control and Optimization **41** (2002), No. 1, 164–180.
- [26] T. Li, *Global exact boundary controllability for first order quasilinear hyperbolic systems*, Discrete Contin. Dyn. Syst. Ser. B 14 (2010), no. 4, 1419–1432.
- [27] T. Li, B. Rao, and Z. Wang, *Exact boundary controllability and observability for first order quasilinear hyperbolic systems with a kind of nonlocal boundary conditions*, Discrete and Continuous Dynamical Systems **28** (2010), No. 1, 243–257.
- [28] A. Majda, *Disappearing Solutions for the Dissipative Wave Equation*, Indiana University Mathematics Journal **24** (1975), No. 12, 1119–1133.
- [29] E. Moulay, W. Perruquetti, *Finite time stability and stabilization of a class of continuous systems*, J. Math. Anal. Appl. 323 (2006), no. 2, 1430–1443.
- [30] E. Moulay, W. Perruquetti, *Finite-time stability and stabilization: state of the art*, Advances in variable structure and sliding mode control, 23–41, Lecture Notes in Control and Inform. Sci., 334, Springer, Berlin, 2006.
- [31] V. Perrollaz, *Initial boundary value problem and asymptotic stabilization of the Camassa-Holm equation on an interval*, J. Funct. Anal. **259** (2010), no. 9, 2333–2365.

- [32] V. Perrollaz, L. Rosier, *Finite-time stabilization of hyperbolic systems over a bounded interval*, in Proceedings of 1st IFAC Workshop on Control of Systems Governed by Partial Differential Equations, Paris, 2013.
- [33] B. de Saint-Venant, *Théorie du mouvement non-permanent des eaux avec applications aux crues des rivières et à l'introduction des marées dans leur lit*, Comptes-rendus de l'Académie des Sciences **73** (1871), 148–154, 237–240.
- [34] M. Slemrod, *A note on complete controllability and stabilizability for linear control systems in Hilbert space*, SIAM J. Control **12** (1974), 500–508.
- [35] J. Smoller, Shock waves and reaction-diffusion equations. Second edition. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 258. Springer-Verlag, New York, 1994.
- [36] Y.-W. Qi, M. Wang, *The global existence and finite time extinction of a quasilinear parabolic equation*, Adv. Differential Equations 4 (1999), no. 5, 731–753.
- [37] C. Z. Xu, G. Sallet, *Exponential stability and transfer functions of processes governed by symmetric hyperbolic systems*, ESAIM: Control, Optimisation and Calculus of Variations **7** (2002), 421–442.

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